This paper was awarded in the II International Competition (1993/94) "First Step to Nobel Prize in Physics" and published in the competition proceedings (*Acta Phys. Pol. A* **88** Supplement, S-49 (1995)). The paper is reproduced here due to kind agreement of the Editorial Board of "Acta Physica Polonica A".

EQUATION OF TIME -- PROBLEM IN ASTRONOMY

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Abstract

The apparent solar motion is not uniform and the length of a solar day is not constant throughout a year. The difference between apparent solar time and mean (regular) solar time is called the equation of time. Two well-known features of our solar system lie at the basis of the periodic irregularities in the solar motion. The angular velocity of the earth relative to the sun varies periodically in the course of a year. The plane of the orbit of the earth is inclined with respect to the equatorial plane. Therefore, the angular velocity of the relative motion has to be projected from the ecliptic onto the equatorial plane before incorporating it into the measurement of time. The mathematical expression of the projection factor for ecliptic angular velocities yields an oscillating function with two periods per year. The difference between the extreme values of the equation of time is about half an hour. The response of the equation of time to a variation of its key parameters is analyzed. In order to visualize factors contributing to the equation of time a model has been constructed which accounts for the elliptical orbit of the earth, the periodically changing angular velocity, and the inclined axis of the earth.
PACS numbers: 95.10.Ce

1 Introduction

1.1. The measurement of time

This paper deals with a problem of the astronomical measurement of time. Let us first introduce some basic definitions. The natural unit of time is the rotation of the earth, that is the apparent daily course of the sun. The length of time between two culminations of the sun is called a *solar day*. The time-system based on this unit is called *apparent solar time*. By this system, localities on a given meridian always have the same time-readings.

A comparison of a sundial with a mechanical clock shows that the solar day has a variable length. Therefore, so-called *mean solar time* is commonly used. This is based on a unit which is defined as the average of a solar day. The mean solar time has been fixed in such a way that it does not deviate too much from the apparent solar time. The deviations between apparent solar time and mean solar time are described by the *equation of time*

$$\text{equation of time} = (\text{apparent solar time}) - (\text{mean solar time}).$$

The derivation, suitable approximations and relevant aspects of the equation of time are discussed in this paper. The derivation does not account for minor effects due to the gravitational fields of the moon and the planets. In principle, therefore, a comparison of the results of such an idealized equation of time with the actual observations can be used to estimate the magnitudes of these effects. Furthermore, parameters of the orbit of the earth, such as its eccentricity, can be verified or calculated. It should be mentioned that the equation of time was very important for navigation in earlier times.

1.2 The periodicity in the solar motion

Two well-known features of our solar system are at the basis of the variations in the apparent motion of the sun:

1. According to Kepler's second law, the angular velocity of the earth relative to the sun varies throughout a year.
2. Equal angles which the sun in its apparent movement goes through in the ecliptic do not correspond to equal angles we measure on the equatorial plane. However, it is these latter angles which are relevant for the measurement of time, since the daily movement of the sun is parallel to the equatorial plane (see Fig. 1).

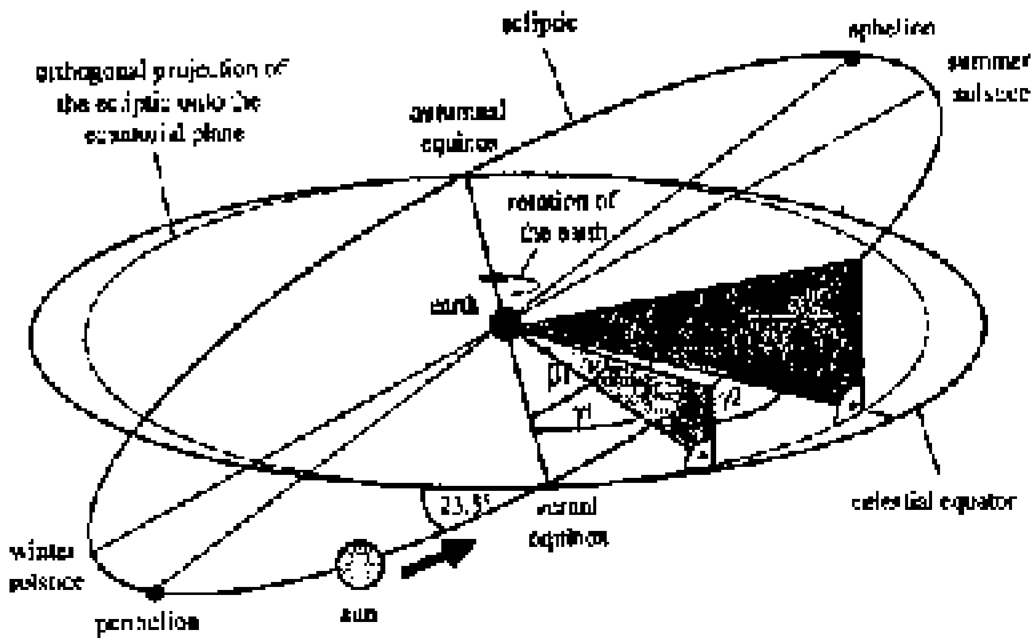

Fig.1. Apparent path of the sun in a geocentric view. At the perihelion, the sun runs faster than at the aphelion. Equal angles β on the plane of the ecliptic do not correspond to equal angles γ on the equatorial plane. In this figure there is $\beta_1 = \beta_2$ but $\gamma_1 < \gamma_2$.

2. The variable angular velocity of the earth

2.1. Kepler's laws

Kepler's first law tells us that all planets are moving in elliptical orbits around the sun, whereby the latter is positioned at one of the two focal points. Kepler's second law -- the so-called law of areas -- describes the velocity of the planets. The area swept out per time interval is constant $(= \mathrm{d}A/\mathrm{d}t)$.

Hence, during the time period *t* the radius vector from the sun to the earth sweeps out an area of

$$t\frac{\mathrm{d}A}{\mathrm{d}t} = t\frac{\pi ab}{T} = \frac{t}{T}\pi ab \tag{1}$$

(a, b -- axes of the ellipse, T -- duration of a revolution).

Let us now derive the angular velocity of the earth as a function of time. The angle covered by the earth after leaving the perihelion is called ``true anomaly", denoted here with *R* (see Fig. 2).

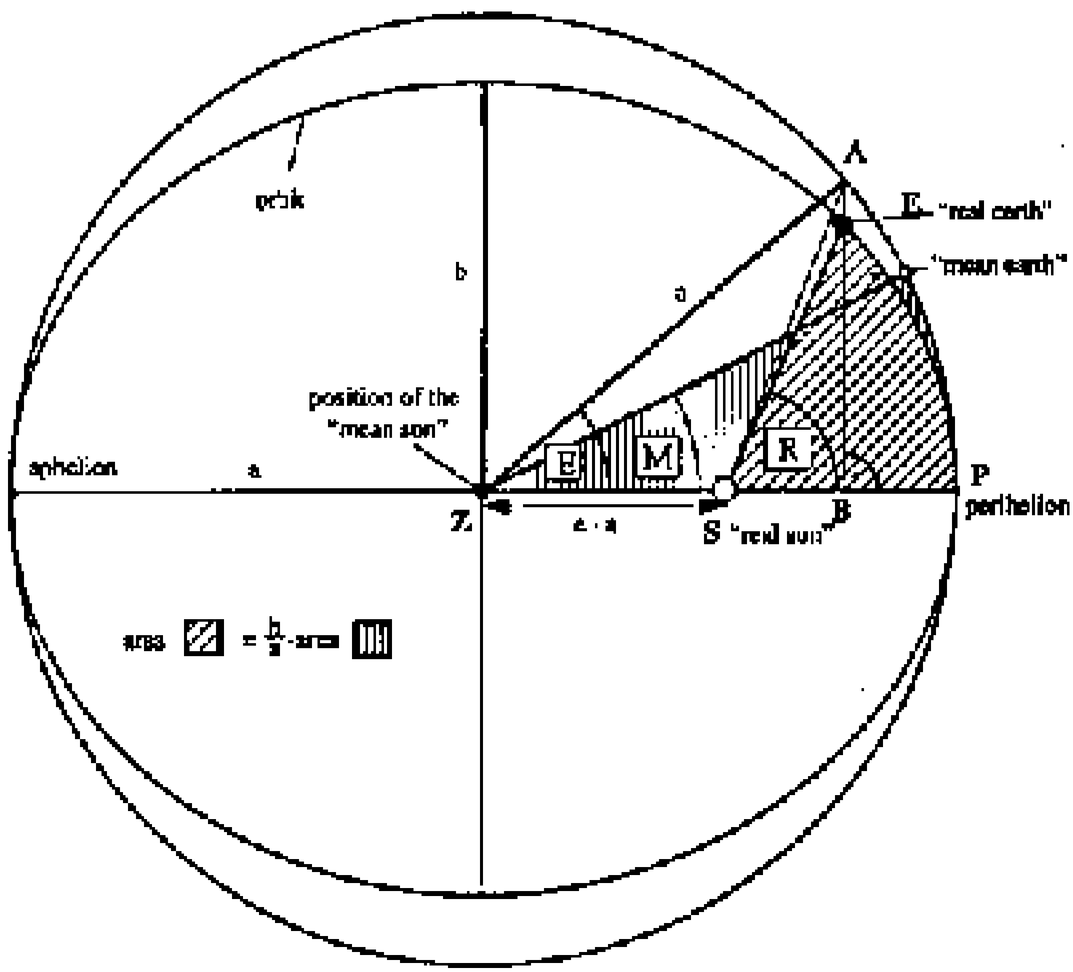

Fig.2. Angles *R*, *M* and *E* at a specific time point. The affinity factor between the elliptical orbit and the circle going through the perihelion and the aphelion is given by *b*/a. As the angular velocity of the ``real earth" and the``mean earth" are both constant, the ratio between the two hatched areas is the same as between the areas of the circle and the ellipse, viz., 1:*b*/a.

Let us imagine a ``mean earth" which has also a revolution time *T* and is running at a constant speed on a circular orbit with the sun at its centre. This ``mean earth" would cover an angle, called ``*mean anomaly"* ($M$), in the same period of time as the true earth covers the angle $R$. In Fig. 2, $M$ is drawn from the centre of the ellipse. The orbit of the ``mean earth" is the circle through the perihelion and the aphelion. The ``mean earth" starts from the perihelion at the same time as the true earth. Since the angular velocity of the ``mean earth" is constant and its revolution lasts one year (T), M satisfies the simple equation

$$M = 2\pi\frac{t}{T}, \qquad (2)$$

where t is the time span after passage through the perihelion. It is very useful to define a third angle as a link between M and R (see Fig. 2). The perpendicular drawn from the position of the true earth (E) onto the major axis intersects the circle, that is the orbit of the ``mean earth", at the point A. The angle PZA is called *eccentric anomaly* (E) and was introduced by Johannes Kepler [1]. It can be used to calculate the area of elliptic sectors.

We get the following relations from Fig. 2

$$\tan R = \frac{\overline{EB}}{\overline{SB}} = \frac{\frac{b}{a}\overline{AB}}{\overline{ZB}-\overline{ZS}} = \frac{\frac{b}{a}a\sin E}{a\cos E - ea} = \frac{b}{a}\frac{\sin E}{\cos E - e} = \frac{\sqrt{1-e^2}\sin E}{\cos E - e}, \qquad (3)$$

$$\cos R = \sqrt{\frac{1}{1+\tan^2 R}} = \sqrt{\frac{(\cos E - e)^2}{(\cos E - e)^2 + (1-e^2)\sin^2 E}} = \frac{\cos E - e}{1 - e\cos E}, \qquad (4)$$

$$\tan\frac{R}{2} = \sqrt{\frac{1-\cos R}{1+\cos R}} = \sqrt{\frac{1 - e\cos E - \cos E + e}{1 - e\cos E + \cos E - e}} = \sqrt{\frac{(1+e)(1-\cos E)}{(1-e)(1+\cos E)}}$$

$$= \sqrt{\frac{1+e}{1-e}}\sqrt{\frac{1-\cos E}{1+\cos E}} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}. \qquad (5)$$

Let us write Eq. (5) with

$$\sqrt{\frac{1-e}{1+e}} = \cos\alpha, \qquad \frac{R}{2} = y, \qquad \frac{E}{2} = x,$$

$$\tan y = \frac{\tan x}{\cos\alpha} \Leftrightarrow y = \arctan\left(\frac{\tan x}{\cos\alpha}\right). \qquad (6)$$

The differentiation of Eq. (6) yields

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos\alpha}{1-\sin^2\alpha\cos^2 x} = \frac{\cos\alpha}{1-\sin^2\alpha\frac{(1+\cos 2x)}{2}}$$

$$= \frac{2\cos\alpha}{1+\cos^2\alpha-\sin^2\alpha\cos 2x} = f(x). \tag{7}$$

Since this expression is a periodic function of x it can be expanded into a Fourier series

$$f(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos\alpha}{1-\sin^2\alpha\cos^2 x}$$

$$= a_0 + a_1\cos x + b_1\sin x + \ldots + a_n\cos nx + b_n\sin nx + \ldots, \tag{8}$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}y = 1 \tag{9}$$

and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\mathrm{d}x, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\mathrm{d}x \text{ for } n>0. \tag{10}$$

Replacing sine and cosine by the complex exponential function yields

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{\cos\alpha\left(e^{inx}+e^{-inx}\right)}{1+\cos^2\alpha-\sin^2\alpha\left(\frac{e^{2ix}+e^{-2ix}}{2}\right)}\mathrm{d}x$$

$$= \frac{\mathrm{i}}{\pi}\oint\frac{\cos\alpha\left(z^{n/2}+z^{-n/2}\right)}{-2z\left(1+\cos^2\alpha\right)+\sin^2\alpha\left(z^2+1\right)}\mathrm{d}z$$

$$= \frac{\mathrm{i}}{\pi}\oint\frac{\cos\alpha\left(z^{n/2}+z^{-n/2}\right)}{\sin^2\alpha\left(z^2-2\frac{(1+\cos^2\alpha)}{\sin^2\alpha}z+1\right)}\mathrm{d}z \qquad \text{with} \qquad z=e^{2ix}. \tag{11}$$

The contour of integration is twice the unit circle. The denominator has the roots

$$N_1 = \left(\frac{1-\cos\alpha}{\sin\alpha}\right)^2 = \tan^2\frac{\alpha}{2},\ N_2 = \left(\frac{1+\cos\alpha}{\sin\alpha}\right)^2 = \cot^2\frac{\alpha}{2}. \tag{12}$$

Now, Eq. (11) can be written in the form

$$a_n = \frac{i}{\pi}\oint \frac{\cos\alpha}{\sin^2\alpha}\frac{\left(z^{n/2}+z^{-n/2}\right)}{\left(z-\tan^2\frac{\alpha}{2}\right)\left(z-\cot^2\frac{\alpha}{2}\right)}\mathrm{d}z$$

$$= \frac{i}{\pi}\oint \frac{\cos\alpha}{\sin^2\alpha}\frac{\left(z^{n/2}+z^{-n/2}\right)}{\tan^2\frac{\alpha}{2}-\cot^2\frac{\alpha}{2}}\left[\frac{1}{\left(z-\tan^2\frac{\alpha}{2}\right)}-\frac{1}{\left(z-\cot^2\frac{\alpha}{2}\right)}\right]\mathrm{d}z$$

$$= -\frac{i}{4\pi}\oint\left(z^{n/2}+z^{-n/2}\right)\left[\frac{1}{\left(z-\tan^2\frac{\alpha}{2}\right)}-\frac{1}{\left(z-\cot^2\frac{\alpha}{2}\right)}\right]\mathrm{d}z. \tag{13}$$

As commonly known, it is

$$\oint z^p\mathrm{d}z = \int_{e^0}^{e^{4i\pi}} z^p\mathrm{d}z = \begin{cases} 0 & \text{for} \quad p \neq -1, \\ 4i\pi & \text{for} \quad p = -1. \end{cases} \tag{14}$$

Therefore, we can develop the fractions of the integrand (13) into convergent series and go on calculating only with terms of the form $g(\alpha)/z$, because all other terms contribute zero to the integral.

$$\frac{1}{z-\tan^2\frac{\alpha}{2}} = \frac{1}{z}\sum_{i=0}^{\infty}\left(\frac{\tan^2\frac{\alpha}{2}}{z}\right)^i, \tag{15}$$

$$-\frac{1}{z-\cot^2\frac{\alpha}{2}} = \tan^2\frac{\alpha}{2}\sum_{i=0}^{\infty}\left(z\tan^2\frac{\alpha}{2}\right)^i. \tag{16}$$

The integrand (13) becomes

$$-\frac{i}{4\pi}\left(z^{n/2}+z^{-n/2}\right)\left[\frac{1}{z}\sum_{i=0}^{\infty}\left(\frac{\tan^2\frac{\alpha}{2}}{z}\right)^i+\tan^2\frac{\alpha}{2}\sum_{i=0}^{\infty}\left(z\tan^2\frac{\alpha}{2}\right)^i\right]. \tag{17}$$

Expressions in $\frac{1}{z}=z^{-1}$ only result if the exponent $\frac{n}{2}$ is an integer. Consequently, the integral is zero if n is odd, that is, a_n vanishes if n is odd. a_n=0 for odd n. If n is even we get the coefficient of $1/z$ in the integrand as the sum of two expressions, each from one of the two series

$$-\frac{i}{4\pi}\frac{1}{z}\left(\tan^n\frac{\alpha}{2}+\tan^2\frac{\alpha}{2}\tan^{-2-n}\frac{\alpha}{2}\right)=-\frac{i}{2\pi}\frac{1}{z}\tan^n\frac{\alpha}{2}. \qquad (18)$$

The integration of this term yields

$$a_n=-\frac{i}{2\pi}\tan^n\frac{\alpha}{2}\int_{e^0}^{e^{2i\pi}}\frac{1}{z}\mathrm{d}z=-\frac{i}{2\pi}\tan^n\frac{\alpha}{2}4i\pi=2\tan^n\frac{\alpha}{2} \qquad (19)$$

$$\Rightarrow a_n=2\tan^n\frac{\alpha}{2}\quad \text{for}\quad \text{even}\quad n>0. \qquad (20)$$

The coefficients b_n can be obtained similarly. If n is odd, b_n disappears. If n is even, one has to write instead of Eq. (18)

$$-\frac{i}{4\pi}\left(\frac{1}{z}\tan^n\frac{\alpha}{2}-\frac{1}{z}\tan^n\frac{\alpha}{2}\right)=0. \qquad (21)$$

Consequently, the Fourier series only consists of cosine terms with an even coefficient in the argument

$$\frac{\mathrm{d}y}{\mathrm{d}x}=1+2\sum_{i=1}^{\infty}\tan^{2i}\frac{\alpha}{2}\cos 2ix, \qquad (22)$$

$$y=x+\sum_{i=1}^{\infty}\tan^{2i}\frac{\alpha}{2}\frac{\sin 2ix}{i}+\text{const.} \qquad (23)$$

In Eq. (5), we have to set

$$\cos\alpha=\sqrt{\frac{1-e}{1+e}}\rightarrow\tan^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{1+\cos\alpha}=\frac{1-\sqrt{\frac{1-e}{1+e}}}{1+\sqrt{\frac{1-e}{1+e}}}=\frac{1-2\sqrt{\frac{1-e}{1+e}}+\frac{1-e}{1+e}}{1-\frac{1-e}{1+e}}$$

$$=\frac{1+e+1-e-2\sqrt{1-e^2}}{1+e-1+e}=\frac{1-\sqrt{1-e^2}}{e}$$

$$\approx\frac{1-\left(1-\frac{e^2}{2}-\frac{e^4}{8}\right)}{e}=\frac{e}{2}+\frac{e^3}{8}. \qquad (24)$$

With $R(E=0)=0$ we get

$$R = E + 2\sum_{i=1}^{\infty}\left(\frac{e}{2}+\frac{e^3}{8}\right)^i \frac{\sin iE}{i}$$

$$\approx E + \left(e + \frac{e^3}{4}\right)\sin E + \frac{e^2}{4}\sin 2E + \frac{e^3}{12}\sin 3E. \qquad (25)$$

The angle E is so useful because the area swept out by the radius vector from the sun to the earth in the time span t can be calculated easily by using the affinity between the circle and the ellipse

$$F = \text{elliptic sector } SEP = \frac{b}{a}(\text{sector } ZAP - \text{triangle } ZAS),$$

$$F = \frac{b}{a}\left(\frac{E}{2\pi}\text{area}_{\text{circle}} - \frac{\overline{ZS}\,\overline{AB}}{2}\right) = \frac{b}{a}\left(\frac{E}{2}a^2 - \frac{aea\sin E}{2}\right)$$

$$= \frac{b}{a}\left(\frac{E}{2}a^2 - a^2\frac{e\sin E}{2}\right) = \frac{ab}{2}(E - e\sin E). \qquad (26)$$

On the other hand, the law of areas implies

$$F = \frac{t}{T}\pi ab, \text{ or with } M = \frac{t}{T}2\pi \text{ one gets } F = \frac{Mab}{2}. \qquad (27)$$

The comparison with Eq. (26) yields

$$E = M + e\sin E \qquad (28)$$

Equation (28) is called Kepler's equation. It is not possible to solve this equation for E in closed form. Therefore, the position of the earth -- given by E -- can only be approximated as a function of time. The ``reversion theorem'' of Lagrange allows us to expand any function in E into a series in e and M. Its basis is an equation in the form

$$z = y + xf(z). \qquad (29)$$

According to Lagrange, any function $g(z)$ can be expanded into a series depending on x and y:

$$g(z) = g(y) + xg'(y)f(y) + \frac{x^2}{2!}\frac{\partial}{\partial y}\left\{g'(y)\,[f(y)]^2\right\}$$

$$+\frac{x^3}{3!}\frac{\partial^2}{\partial y^2}\left\{g'(y)[f(y)]^3\right\}+\ldots \quad (30)$$

In the case of Kepler's equation (28), we set

$$z=E, \quad y=M, \quad x=e, \quad f(z)=\sin E, \quad f(y)=\sin M, \quad g(z)=z. \quad (31)$$

E turns out as

$$E=M+e\sin M+\frac{e^2}{2}\frac{\partial}{\partial M}(\sin^2 M)+\frac{e^3}{6}\frac{\partial^2}{\partial M^2}(\sin^3 M)+\ldots$$

$$=M+e\sin M+\frac{e^2}{2}\sin 2M+\frac{e^3}{8}(3\sin 3M-\sin M)+\ldots \quad (32)$$

To be able to express R in Eq. (25) as a function of M, we also have to expand the sine functions of E, $2E$ and $3E$ as far as needed

$$\sin E=\frac{E-M}{e}=\sin M+\frac{e}{2}\sin 2M+\frac{e^2}{8}(3\sin 3M-\sin M)+\ldots \quad (33)$$

$$\sin 2E=\sin 2M+e\sin M\, 2\cos 2M+\ldots$$

$$=\sin 2M+e(\sin 3M-\sin M)+\ldots \quad (34)$$

$$\sin 3E=\sin 3M+\ldots \quad (35)$$

Replacing the functions in E of Eq. (25) with these formulas in M, we obtain

$$R\approx E+\left(e+\frac{e^3}{4}\right)\sin E+\frac{e^2}{4}\sin 2E+\frac{e^3}{12}\sin 3E$$

$$=M+\left(2e+\frac{e^3}{4}\right)\sin E+\frac{e^2}{4}\sin 2E+\frac{e^3}{12}\sin 3E$$

$$\approx M+2e\sin M+\frac{5}{4}e^2\sin 2M+e^3\left(-\frac{1}{4}\sin M+\frac{13}{12}\sin 3M\right)+e^4\ldots \quad (36)$$

In this series, we have only taken into account terms up to third power of e. (This approximation is already very accurate, for e is about 0.0167.) The angular velocity of the earth can now be obtained as the first derivative of R with respect to time

$$\omega(t) = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}M}\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{2\pi}{T}\frac{\mathrm{d}R}{\mathrm{d}M}$$

$$\approx \frac{2\pi}{T}\left[1 + 2e\cos M + \frac{5}{2}e^2\cos 2M + e^3\left(-\frac{1}{4}\cos M + \frac{13}{4}\cos 3M\right)\right]. \qquad (37)$$

This function is plotted in Fig. 3. It is mainly determined by the first variable term of the series $(2e\cos M)$. The factor $2\pi/T$ is the mean angular velocity. The deviations amount to about $+3.5$ % ($\approx 2e$).

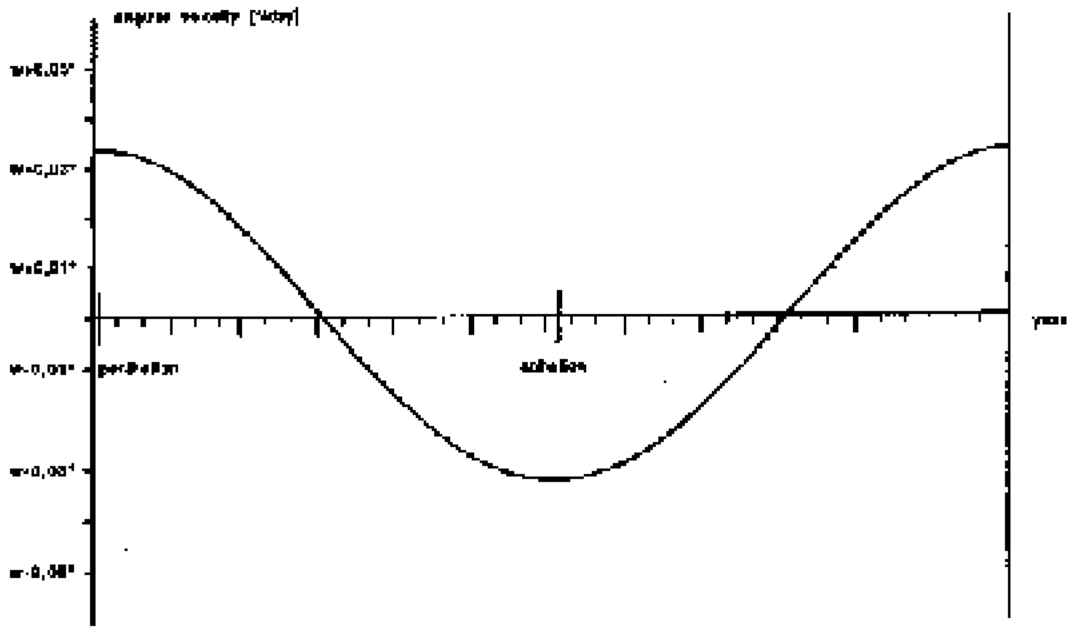

Fig.3. The angular velocity of the earth as a function of time. On average it is about $2\pi/365.25$ per day or $360/365.25$ °/day = 0.986 °/day.

3. The inclined plane of the ecliptic

3.1. The earth in space

Since the equation of time is to be examined during a year, the earth can be supposed to always have the same direction in space, i.e., we will not take into account precession and nutation.

We can describe the direction of the axis of the earth with two angles. The first is the angle between the axis and the norm of the orbit (ε). The second is the angle P formed by the major axis of the orbit and the projection of the axis of the earth onto the plane of the orbit (see Fig. 4). At present, ε measures about $23.45°$, *P* is about $12.25°$. (*P* is also the angle which is covered by the earth between the beginning of winter (21st December) and the arrival of the earth at perihelion (2nd January).)

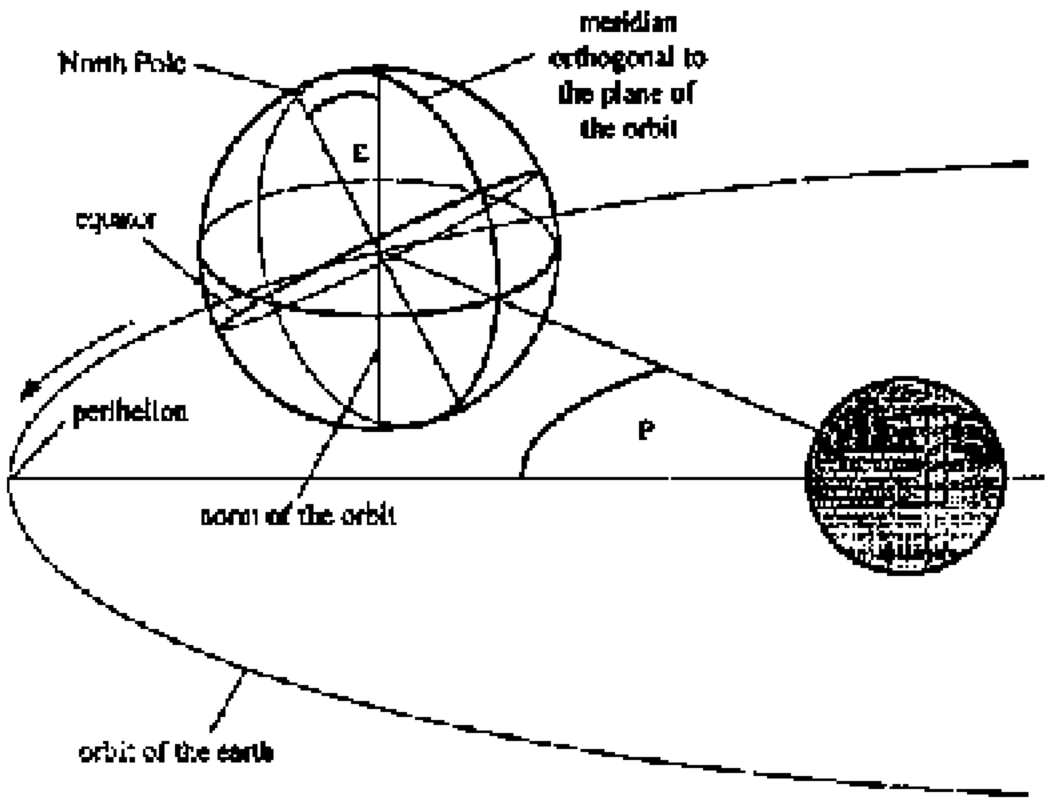

Fig.4. The angle *P* and the inclination of the axis of the earth at the winter solstice.

3.2. The projection of ecliptic angles onto the equatorial plane

The angles which the sun appears to cover relative to the earth are equal to those the earth actually covers relative to the sun. Since the corresponding angles parallel to the equatorial plane are needed for the measurement of time, the ecliptical angles have to be projected onto the equatorial plane. This results in angle widening or shortening. For an infinitesimally small angle the projection factor (the deformation factor) is determined by the angle parameters φ and ε in Fig. 5, which represents the geocentric view. φ is measured from the winter solstice.

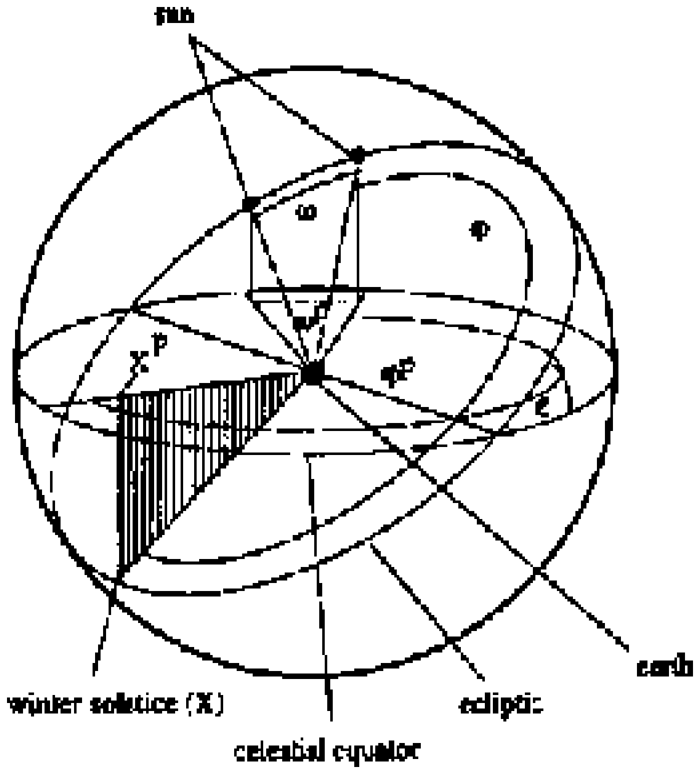

Fig.5. Geocentric view of the projection. In a short time interval, the sun has covered the angle ω. ω^p is the orthogonal projection of ω.

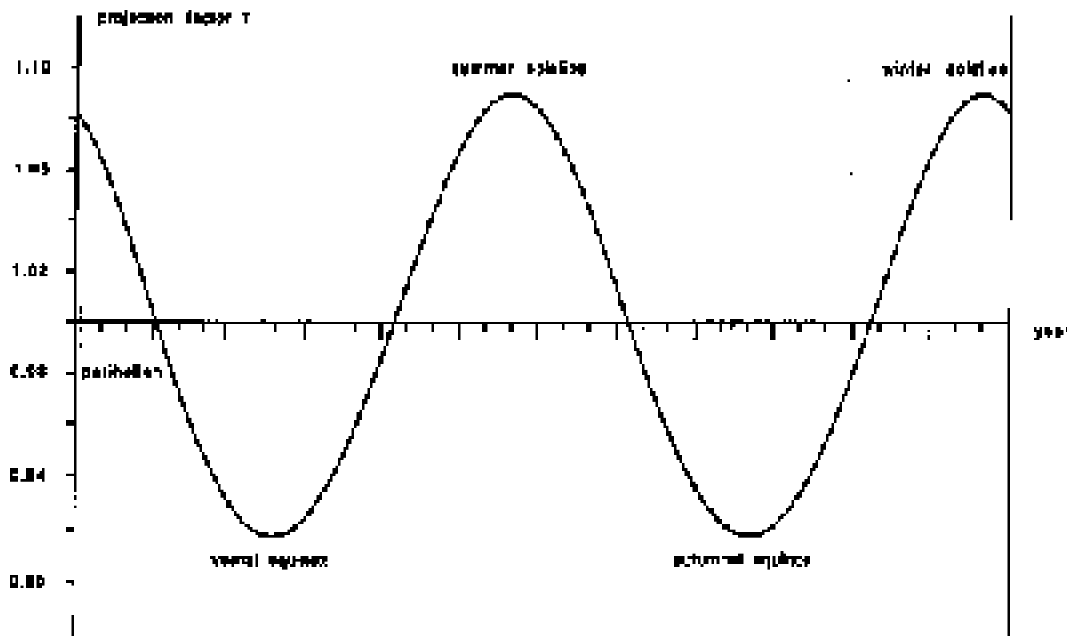

Fig.6. The projection factor as a function of time. The maxima are at the solstices, the minima at the equinoxes.

We get the following relations from Fig. 5:

$$\tan\varphi = \cos\varepsilon\tan\varphi^{\mathrm{P}}, \qquad \tan(\varphi+\omega) = \cos\varepsilon\tan(\varphi^{\mathrm{P}}+\varphi^{\mathrm{P}})\,. \tag{38}$$

The projection factor *f* for a small angle can be calculated as a function of φ and ε :

$$f(\varphi) = \lim_{\omega\to\infty}\left(\frac{\omega^{\mathrm{P}}}{\omega}\right) = \lim_{\omega\to 0}\left\{\frac{\arctan\left[\frac{\tan(\varphi+\omega)}{\cos\varepsilon}\right] - \arctan\left(\frac{\tan\varphi}{\cos\varepsilon}\right)}{\omega}\right\}. \tag{39}$$

Using the series (22) derived in Sec. 2.1 ($\varepsilon = \alpha$), we get an approximation which converges very quickly

$$f(\varphi) = f(R+P) = 1 + 2\tan^2\frac{\varepsilon}{2}\cos 2(R+P)$$

$$+2\tan^4\frac{\varepsilon}{2}\cos 4(R+P) + 2\tan^6\frac{\varepsilon}{2}\cos 6(R+P) + \ldots \tag{40}$$

The projection factor *f* is plotted as a function of time in Fig. 6. This graph resembles a cosine function since it is mainly determined by the first variable term of the approximation. The amplitude of the variation $(2\tan^2\frac{\varepsilon}{2})$ amounts to $8\frac{1}{2}$ % and, thus, is greater than that of the angular velocity (compare with Fig. 3).

The extremes are situated at the beginning of the seasons. At the summer and winter solstice, the sun reaches, respectively, its highest and lowest position. Here, an ecliptic angle is stretched maximally. At the vernal and autumnal equinox, the sun stands vertically above the equator. Here, ecliptic angles are shortened maximally.

4. The calculation of the equation of time

4.1. Derivation of a more accurate approximation

In order to calculate the equation of time we need the projection of the true anomaly R^{P} as a function of time

$$R^{\mathrm{P}} = \int \mathrm{d}R^{\mathrm{P}} = \int f(R+P)\mathrm{d}R = \int\left[1+2\sum_{i=1}^{\infty}\tan^{2i}\frac{\varepsilon}{2}\cos 2i(R+P)\right]\mathrm{d}R \tag{41}$$

$$= R + \sum_{i=1}^{\infty}\frac{1}{i}\tan^{2i}\frac{\varepsilon}{2}\sin 2i(R+P) + \mathrm{const.} \tag{42}$$

The constant term will be determined later. Let us replace *R* by *M* with Eq. (36)

$$R_{\odot}^{\circ} = M + 2e\sin M + \frac{5}{4}e^2\sin 2M + e^3\left(-\frac{1}{4}\sin M + \frac{13}{12}\sin 3M\right)\ldots \tag{43}$$

Thus, R^P becomes a function of M and hence of time.

$$R^P = M + 2e\sin M + \frac{5}{4}e^2\sin 2M + e^3\left(-\frac{1}{4}\sin M + \frac{13}{12}\sin 3M\right) + \ldots$$

$$+\tan^2\frac{\varepsilon}{2}\sin\left\{2(M+P) + 2\left[2e\sin M + \frac{5}{4}e^2\sin 2M\right.\right.$$

$$\left.\left.+e^3\left(-\frac{1}{4}\sin M + \frac{13}{12}\sin 3M\right) + \ldots\right]\right\}$$

$$+\frac{1}{2}\tan^4\frac{\varepsilon}{2}\sin\left\{4(M+P) + 4\left[2e\sin M + \frac{5}{4}e^2\sin 2M\right.\right.$$

$$\left.\left.+e^3\left(-\frac{1}{4}\sin M + \frac{13}{12}\sin 3M\right) + \ldots\right]\right\}$$

$$+\frac{1}{3}\tan^6\frac{\varepsilon}{2}\sin\left\{6(M+P) + 6\left[2e\sin M + \frac{5}{4}e^2\sin 2M\right.\right.$$

$$\left.\left.+e^3\left(-\frac{1}{4}\sin M + \frac{13}{12}\sin 3M\right) + \ldots\right]\right\} + \ldots + \text{const.} \tag{44}$$

The expansion of the sine functions yields

$$R^P \approx M + \tan^2\frac{\varepsilon}{2}(1-4e^2)\sin 2(M+P) + 2e\sin M$$

$$-2e\tan^2\frac{\varepsilon}{2}\sin(M+2P) + 2e\tan^2\frac{\varepsilon}{2}\sin(3M+2P)$$

$$+\frac{1}{2}\tan^4\frac{\varepsilon}{2}\sin 4(M+P)+\frac{5}{4}e^2\sin 2M-2e\tan^4\frac{\varepsilon}{2}\sin(3M+4P)$$

$$+2e\tan^4\frac{\varepsilon}{2}\sin(5M+4P)+\frac{13}{4}e^2\tan^2\frac{\varepsilon}{2}\sin(4M+2P)$$

$$+\frac{1}{3}\tan^6\frac{\varepsilon}{2}\sin 6(M+P)+\text{const.} \qquad (45)$$

The equation of time is defined by

$$\begin{aligned}\text{equation of time} &= \text{“true equatorial sun angle”} \\ &\quad - \text{“mean equatorial sun angle” (geocentric view)} \\ &= -(\text{true projected anomaly} - \text{mean anomaly}) \\ &\quad \text{(heliocentric view)} \\ &= M - R^p. \end{aligned} \qquad (46)$$

Let us now determine the constant of integration. A commonly used definition implies that the ``mean sun" arrives at the vernal equinox at the same time as a ``dynamic sun" that runs in the ecliptic at a constant speed and leaves the perihelion at the same time as the real sun (see [2]). Because of this definition the angles of the two regularly running suns to the vernal equinox are always equal. Consequently, the angle P between the perihelion and the direction earth-winter solstice in Fig. 7 is equal to the angle of the ``dynamic sun" on the equator to the projected winter solstice direction. The constant of integration can now be determined if we examine the passage through the perihelion (see Fig. 7):

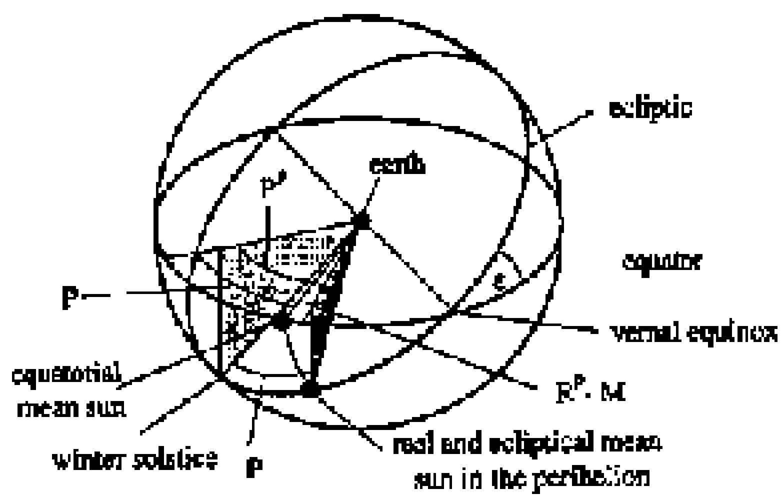

Fig.7. The determination of the constant of integration $R^p - M = P^p - P$.

$$M - R^p = P - P^p = P - \int_0^P f(\varphi)\mathrm{d}\varphi$$

$$= P - \int_0^P \left(1 + 2\sum_{i=1}^{\infty} \tan^{2i}\frac{\varepsilon}{2}\cos 2i\varphi\right)\mathrm{d}\varphi = -\sum_{i=1}^{\infty}\frac{1}{i}\tan^{2i}\frac{\varepsilon}{2}\sin 2iP. \qquad (47)$$

If we set t=0 in Eq. (44) we will find that the constant of integration vanishes. Now we can calculate the coefficients in Eq. (45) with $\varepsilon = 23.45^\circ$ and e=0.0167. Finally we get

$$\text{equation of time} = M - R^p \approx -591.7\sin 2(M+P) - 459.6\sin M$$

$$+19.8\sin(M+2P) - 19.8\sin(3M+2P) - 12.8\sin 4(M+P)$$

$$-4.8\sin 2M + 0.9\sin(3M+4P) - 0.9\sin(5M+4P)$$

$$-0.5\sin(4M+2P) - 0.4\sin 6(M+P) \quad [\mathrm{s}]. \qquad (48)$$

At present, the angle P measures about 12.25°.

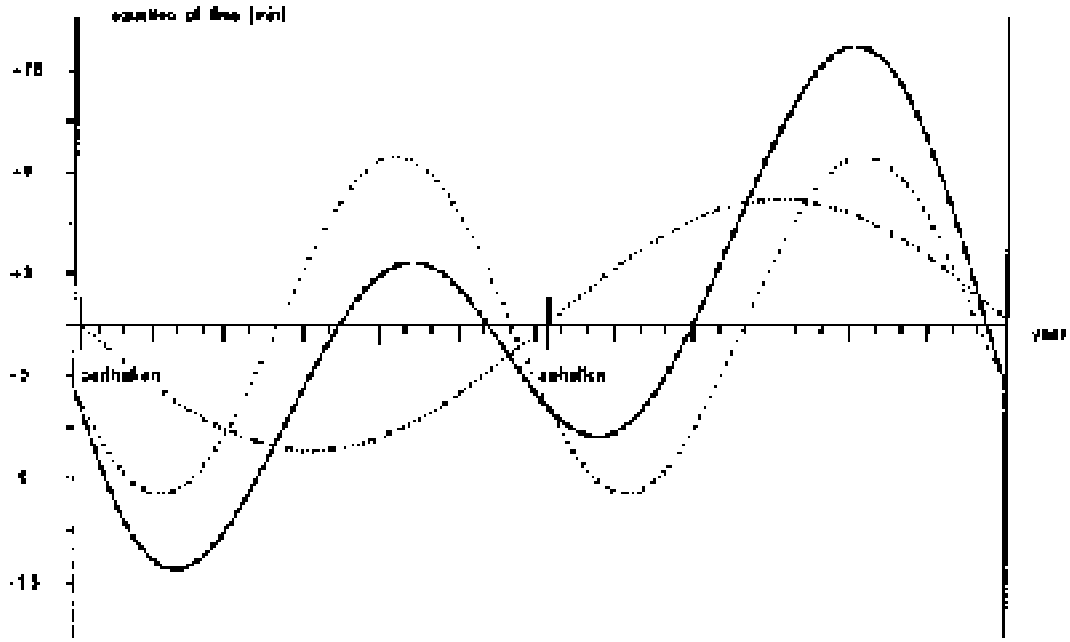

Fig.8. The equation of time with its two main terms. The difference between the maximum in October (≈ 16 min) and the minimum in February (≈ -14 min) is about half an hour.

Figure 8 shows the equation of time for a whole period of one year. In winter, the value of the equation of time decreases mostly because the angular velocity of the earth and the projection factor reach their maximum. The opposite holds true between the passage through the aphelion and the beginning of autumn. Afterwards, the equation of time itself reaches an extreme value. In the time span between those extrema, especially in summer, the lower angular velocity and the greater projection factor compensate each other.

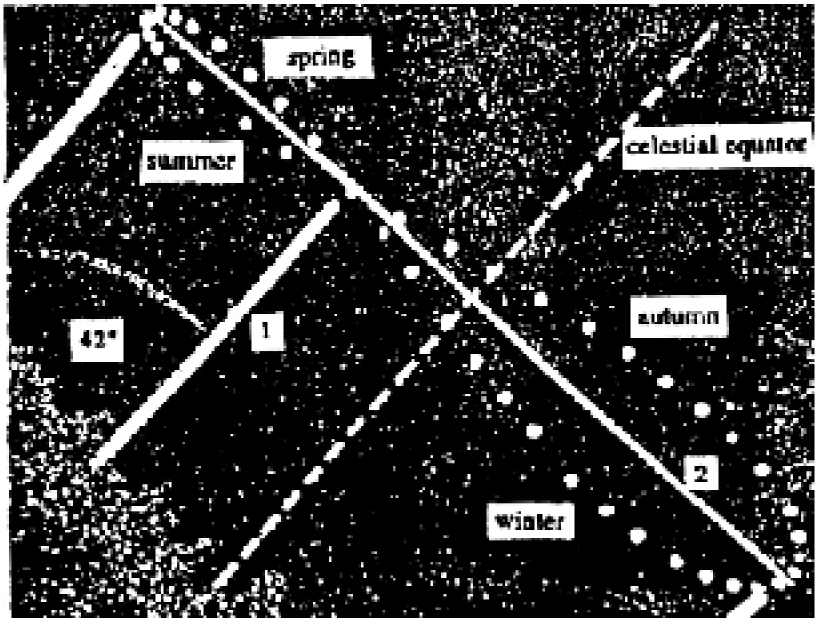

Fig.9. The analemma (after [2]).

Figure 9 (taken from [2]) shows the observed phenomenon of the equation of time. In intervals of ten days, a picture of the sky was taken at 8.14 (mean solar time). If the sun moved at a constant speed through the sky, all pictures of the sun would lay on a straight line that would be perpendicular to the daily path of the sun. In winter and in summer, the sun has not reached as far as one would expect (it is below the ``average line'' (2)). Therefore, the value of the equation of time is negative (see Fig. 8). On the other hand, the sun advances faster from April to June and at the end of the year (it is above the ``average line'') and the value of the equation of time is positive. These variations produce the form of a stretched and inclined ``eight'' in the picture.

4.2. The equation of time as a function of its parameters

The equation of time is determined by the following parameters:

- the eccentricity of the orbit of the earth
- the angle between the ecliptic and the equatorial planes
- the angle P between the winter solstice and the perihelion relative to the sun
  or: the time span Δt from the beginning of winter to the passage through the perihelion

The last two parameters change gradually by nutation and precession. It is therefore interesting to examine the influence of each parameter. Figures 10-12 show how the equation of time changes when one parameter is varied.

1. parameter: the eccentricity. If e=0 a regular variation results that is caused by the inclination of the ecliptic plane. The deviations of the apparent solar time from the mean solar time increase with growing

e in winter and autumn. Thus, the yearly variation becomes dominant. Since at the perihelion and aphelion the equation of time is only a function of the ecliptic inclination and the angle P, all plots have the same value at these two points.

2. parameter: the inclination of the ecliptic. $\varepsilon = 0$ yields a plot which is symmetric to the passage through the aphelion. The greater ε the more dominant the variation with a period of half a year. All plots have four common points at the beginning of each season, for the equation of time depends only on the two other parameters there (eccentricity and P). As the projection from the ecliptic plane onto the equatorial plane does not change the polar angle relative to the winter solstice, ε does not influence the value of the equation of time at the beginning of a season.

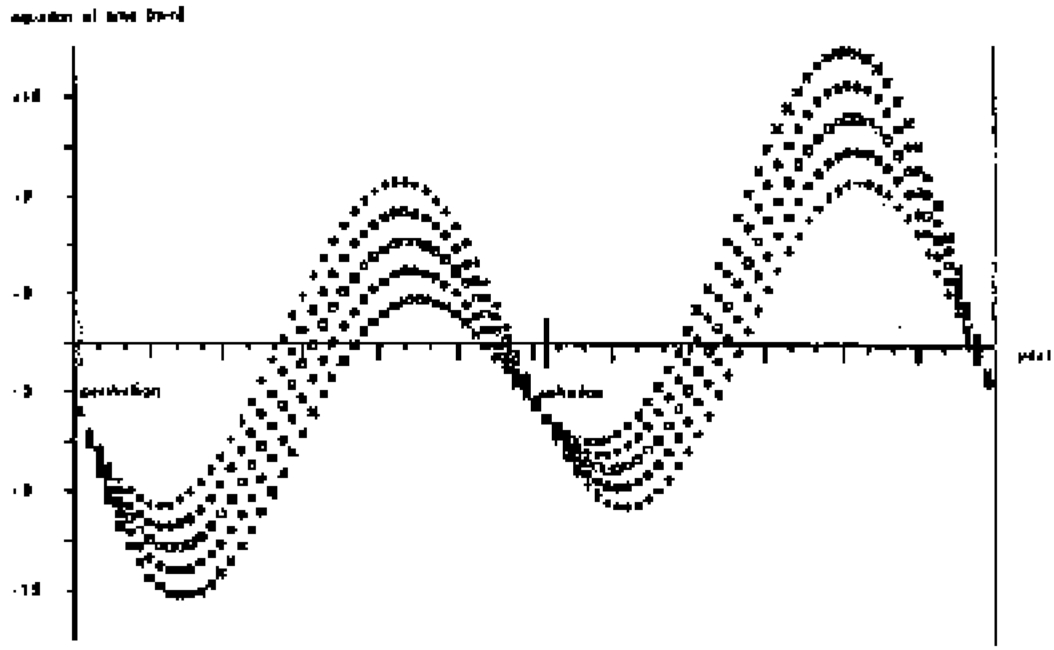

Fig.10. 1. parameter: the eccentricity. +++ e=0.000, ∘ ∘ ∘ $e = 0.005$, □ □ □ $e = 0.010$, • • • $e = 0.015$, x x x e=0.020.