THE MEAN VELOCITY DISTRIBUTION NEAR THE PEAK OF THE REYNOLDS SHEAR STRESS, EXTENDING ALSO TO THE BUFFER REGION

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Abstract: An expression is derived for the mean velocity distribution in pipe and channel flows near the position of the maximum Reynolds shear stress, y_m. This expression agrees well with measurements in a significant region on both sides of y_m, extending to the buffer region on the one hand and almost all the way to the centerline of the flow on the other.

Key words: mean velocity, peak Reynolds shear stress, buffer region.

1. INTRODUCTION

Close to the surface in wall-bounded flows such as pipes, channels and boundary layers, the mean velocity varies linearly with wall-normal distance [1]. Further away from the surface, the traditional understanding has been that the variation is logarithmic [2]. Recent work on this same issue [3] has proposed power law variation as more appropriate (though the suggestion of an empirical power-law fit goes back to Prandtl and his students). Even further out in the flow, the so-called wake function [4] is thought to codify experimental data.

This article does not elucidate the work of Refs. [1-4] directly. We merely use a simple tool to construct an explicit expression for the distribution of the mean velocity near the position of maximum Reynolds shear stress, y_m. The key idea is the logarithmic power series expansion around y_m. Such an expansion is usually suitable when there is some long-range interaction in

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the problem, as seems to be true for turbulent flows in general—wall-bounded flows in particular. The validity of the expression so derived extends, on one side of y_m, to the so-called buffer region that exists between the end of the linear region and the beginning of the log/power region. The buffer region is of great importance because of its participation in the turbulence generation mechanism, yet there exist no explicit expressions for the velocity distribution in this region—especially those based on broadly applicable physical principles. On the other side of y_m, the expression seems to be valid almost all the way to the centerline of the flow.

2. ANALYSIS

Let us start from the exact equation valid for pipes and channels

$$-\langle uv\rangle^+ = -dU^+/dy^+ + (1 - y^+/R^+), \qquad (1)$$

in which we have used the standard notation: u and v are velocity fluctuations in streamwise and wall-normal directions x and y respectively, U(y) is the mean velocity in the direction x, R is the pipe radius or the channel half-height, and the suffix + indicates normalization by wall variables u_τ and ν, which represent, respectively, the friction velocity and the fluid viscosity. Elementary consideration show that the turbulent stress term $-\langle uv\rangle^+$ increases cubically with y very close to the wall; it changes rapidly into a different form that has not been studied carefully so far before attaining a maximum value in the flow; it subsequently drops off to zero as the flow centreline is approached further outwards [5]. The position of the maximum in the Reynolds shear stress, y_m, is empirically known [6] to obey

$$y_m^+ \approx 1.87\, R_\tau^{1/2}, \qquad (2)$$

where the Reynolds number $R_\tau \equiv u_\tau R/\nu$. This fit has been proposed by others as well [7]. Though the multiplicative constant is slightly different in each work, this ambiguity merely reflects the uncertainty associated with the identification of y_m from measured data and is not fundamental. It should be stressed that the distribution of $-\langle uv\rangle^+$ has been obtained by numerically differentiating the measured mean velocity distribution and using (1), and so is not dependent on the inaccuracies that usually plague the Reynolds shear stress measurements.

Let us expand $-\langle uv\rangle^+$ around y_m^+. We had undertaken this exercise already in [6] but had not appreciated the importance of expanding $-\langle uv\rangle^+$

in terms of the logarithm of the distance from y_m^+. It now seems to us that this is the appropriate expansion to make, considering that the Reynolds shear stress varies slowly in the region around its peak. In general, expansions in logarithmic variables are appropriate whenever long-range effects are present, as is the case in wall-bounded flows. We may then write

$$-\langle uv\rangle^+ = c[1-\alpha_1\{\ln(y^+/y_m^+)\}^2 + \ldots + \alpha_n\{\ln(y^+/y_m^+)\}^n + \ldots]. \quad (3)$$

Here, the unknown constants $\alpha_1 \ldots \alpha_n$ are thought to be independent of the Reynolds number, at least when it is high enough. The fit works very well for all Reynolds numbers shown in figure 1, roughly for $y^+ \geq 10$. This region more or less borders the buffer region.

Substituting (3) in (1), and retaining only the first two terms in the expansion (3), we obtain

$$U^+ = \text{const} + y^+ g(y^+) - (y^{+2}/2R^+), \quad (4)$$

where

$$g(y^+) = a_0 + a_1\,[\ln(y^+/y_1)]^2, \quad (5)$$

$$\text{with } a_0 = 1 - c + c\alpha_1,\ a_1 = c\alpha_1,\ y_1 = e y_m^+. \quad (6)$$

The expression (4) is technically not expected to be valid all the way to the wall (see figure 1), but we can be somewhat rough and impose the no slip condition $U^+ = 0$ at $y^+ = 0$ to obtain

$$U^+ = y^+[g(y^+) - y^+/2d^+], \quad (7)$$

where $d^+ = 2R^+$. In order to compare the last equation directly with experimental data, it is useful to rewrite it in the form

$$U^+/y^+ + y^+/d^+ = g(y^+) = a_0 + a_1\,[\ln(y^+/y_1)]^2. \quad (8)$$

If the present considerations are valid, the left hand side of (8) must show a parabolic variation with respect to y^+ in logarithmic coordinates.

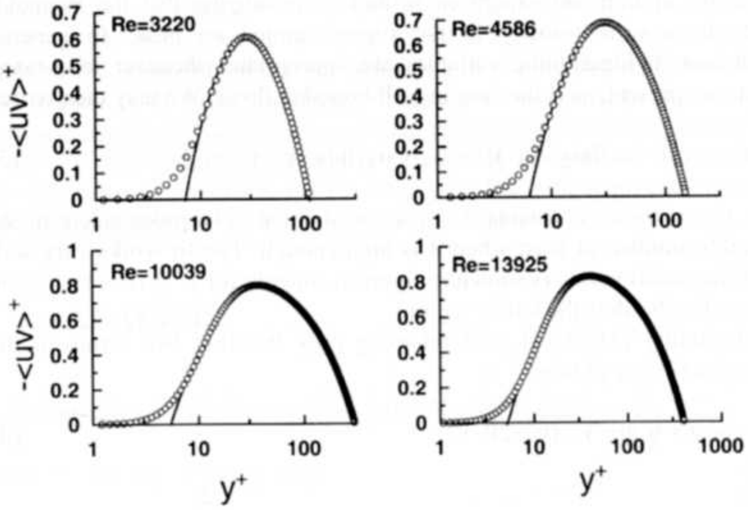

Figure 1. Plots of the Reynolds shear stress from the direct numerical simulations of a channel flow [8], for four different Reynolds numbers, Re, based on the bulk mean velocity and the width of the channel. The data have been fitted by the two term expansion of (3). The fit is very good for $y^+ > 10$.

3. COMPARISON WITH MEASUREMENTS

We show in figures 2 and 3 the recent Princeton data [9] for two Reynolds numbers. The solid parabolas are drawn in order to compare the data with equation (8) in the semi-logarithmical scales. The agreement with the data is excellent almost all the way to y^+ of the order 10 towards the wall, and to y^+ of the order 1000 or more outwards—in fact, almost all the way to the centerline.

4. CONCLUSIONS

In the traditional picture, the Reynolds shear stress attains a constant value of unity, this being the fundamental factor leading to the logarithmic law. That one can identify a maximum value from the distribution of $-\langle uv\rangle^+$ is a reflection that the expected constancy does not obtain at least up to the Reynolds number for which (2) holds. It may be that the relation that holds

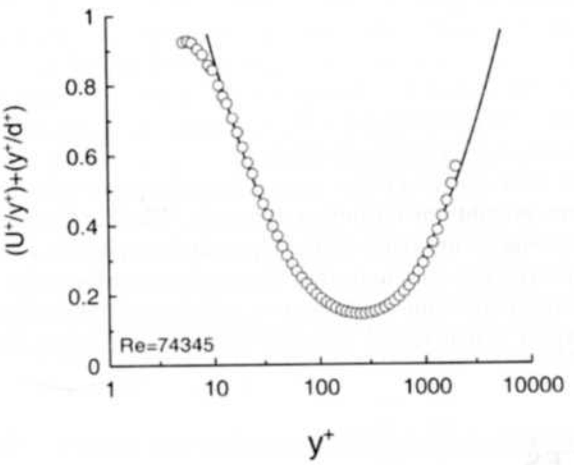

Figure 2. A plot of $U^+/y^+ + y^+/d^+$ against y^+ in semi-log scales (circles), for Re = 74,345, where Re is based on the mean velocity and the pipe diameter. The data are from [9]. The solid parabola indicates correspondence to equation (8).

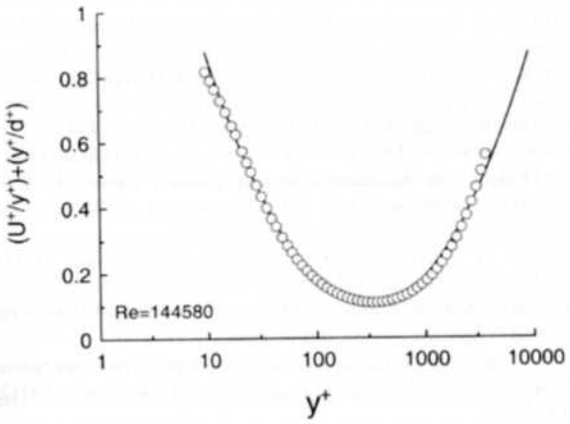

Figure 3. As in figure 2, but for Re = 144,580. Again, the data are from [9].

for "low" Reynolds numbers—in which case the present considerations hold only that range of Reynolds numbers. This possibility is equivalent to the scenario in which y_m^+ remains unchanged beyond a certain Reynolds number. On the other hand, if a maximum can indeed be identified at all Reynolds numbers, this feature has to be taken into account in some way. Such considerations were the subject of [6].

We ourselves view equation (8) as a good fit to the mean velocity data in the buffer region, possibly much further outwards. Whether the proposal is fundamental depends on the status of the logarithmic expansion (3) that we have used. At present, it is hard to resolve the question satisfactorily.

The analysis is strictly valid for only pipe and channel flows (because of (1)), but we expect that it would be valid for constant-pressure boundary layers as well.

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