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# SAM EDWARDS AND THE TURBULENCE THEORY

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**Abstract**

We provide a brief assessment of the contributions of Sir Sam Edwards to field-theoretic methods in the statistical theory of turbulent fluid dynamics, and connect those contributions to later developments in the subject.

## 4.1 Introduction

The closure problem in hydrodynamic turbulence is notorious for its difficulty (see, *e.g.*, Monin and Yaglom 1971). Expansions of high-order moments in terms of powers of Reynolds number do not converge; truncation of the hierarchy of moments or cumulants beyond a certain order yield unrealizable results such as negative energy. Sam Edwards was immersed in this problem in the mid-sixties. In his first paper on the subject, 'Theoretical dynamics of homogeneous turbulence,' *J. Fluid Mech.* **18**, 239 (1964) he already stated its essential difficulty:

> 'Many problems in theoretical physics can be expressed in terms of functional differential equations, but turbulence is an exceptional problem in that there is in the limit of large Reynolds number no external parameter which can be used as a basis of an expansion technique. In the language of quantum field theory it is a problem of infinitely strong coupling constant.'

The turbulence problem perhaps no longer appears to be as 'exceptional' as it once did, for other important strong-coupling problems have since been faced in theoretical physics. Some of these, such as color confinement in quantum chromodynamics, are still with us; others, such as critical phenomena in three space dimensions, have been successfully solved.[1] Experience has taught us that each

[1] We have used the word 'solved' to indicate that critical scaling exponents have been calculated by several methods, such as Borel-resummed $\epsilon$-expansion, high-temperature series, and Monte Carlo simulation, and that the results agree to several significant digits [for recent discussions, see Guida and Zinn-Justin (1998), and Pelissetto and Vicari (2002)]. Mathematical



such strong-coupling problem stands on its own, and no general and encompassing method is available, or perhaps likely to be found, to solve them all. In the case of critical phenomena, it was discovered that there were 'hidden' small parameters, such as the deviation of space dimension $d$ from an upper-critical dimension $d_c$ (often four), namely $\epsilon = d_c - d$, or an inverse number of components of the order parameter, $1/N$. These parameters were made the basis of successful perturbative calculations even for $\epsilon = 1$ or $N = 1$, especially when augmented with Padé or Borel resummation techniques (Wilson and Fisher 1972; Ma 1976; Fisher 1998). It has also been possible to develop successful nonperturbative numerical schemes to calculate critical scaling exponents, *e.g.*, by fast Monte Carlo algorithms. [See Guida and Zinn-Justin (1998), and Pelissetto and Vicari (2002) for a survey of current results.]

None of these methods that enabled break-through successes in the theory of critical phenomena has yielded results of comparable significance in understanding or predicting turbulent flows. Nevertheless, considerable progress has been made. Edwards himself was a pioneer in the application of quantum field-theory tools to turbulence. In his paper cited above (Edwards 1964), he developed a self-consistent expansion method which, in his own words, was 'based on the internal properties of the system.' His methods, as well as the related earlier work of Kraichnan (1959, 1961) and Wyld (1961), have yielded some important insights. Recently, perturbative techniques have scored a very significant success in calculating turbulent scaling exponents in a simplified model of a white-noise advected passive scalar (for a review, see Falkovich *et al.* 2001). It is our purpose in this chapter to review the contribution of Sam Edwards to developing field-theoretic methods in turbulence theory, and to summarize some recent progress and hopes for the future. In Section 4.2 we shall take a quick tour through Edwards' classic paper and point out some of its significant results that have played a role in later developments. In particular, in Section 4.3, we shall discuss the recent progress in calculating anomalous scaling exponents in the Kraichnan model for passive scalars by perturbative field-theoretic methods. In Section 4.4 we offer some prognosis for the much harder problem of Navier-Stokes turbulence. The paper concludes with a brief summary and perspective in Section 4.5.

## 4.2 Contributions of Edwards

Edwards (1964) contains many noteworthy aspects but we shall focus on those that seem to us most important in view of our present understanding of the subject, and on those that have played some part in later developments. Other

physicists regard aspects of the problem as open: *e.g.*, no one has yet constructed by rigorous mathematics a strong-coupling fixed point of the renormalization group for a realistic, short-ranged model in three dimensions, although the numerical evidence is that it exists and the scaling properties in its vicinity are understood by the $\epsilon$-expansion. Various non-universal quantities of significant interest, such as critical temperatures, cannot yet be readily calculated for real physical systems found in nature or realized in laboratories. Engineers might in general regard a problem as open if it is understood only to this degree of detail.

scientists revisiting the paper will certainly uncover riches left uncounted and undescribed in our summary. Everyone interested in the subject of turbulence is thus warmly encouraged to read the paper for himself and discover hidden vistas unremarked upon by us.

The problem that Edwards considers is the idealized case of homogeneous isotropic turbulence maintained statistically steady by a stochastic fluctuating force, which acts as the random source of energy:

$$\frac{\partial \mathbf{U}}{\partial t} = \nu \boldsymbol{\nabla}^2 \mathbf{U} - (\mathbf{U} \cdot \boldsymbol{\nabla})\mathbf{U} - \boldsymbol{\nabla} p + F. \tag{4.1}$$

Here $\mathbf{U}$ is the fluid velocity, with pressure $p$ determined to enforce incompressibility $\boldsymbol{\nabla} \cdot \mathbf{U} = 0$. The stirring force $F$ chosen by Edwards is a Gaussian random field with zero mean and covariance given by

$$\langle F_i(\mathbf{r},t) F_j(\mathbf{r}',t')\rangle = g_{ij}(\mathbf{r}-\mathbf{r}', t-t'). \tag{4.2}$$

Because the energy input is stochastic, it is only possible to seek statistical information about the system. Edwards thus focuses on the multivariate probability density for Fourier amplitudes. The first step in his paper is the derivation of the Liouville equation for the probability density. It resembles Hopf's (1952) functional equation with which it shares the property of linearity. On the way to deriving this equation, Edwards obtains the now well-known relation for the mean energy input of a Gaussian white-noise force, for which the spatial Fourier transform of the noise covariance $g_{\mathbf{k}}(t-t')$ is equal to $h_{\mathbf{k}}\delta(t-t')$. The mean energy input is just the integral over wavenumber of the forcing spectrum $h_{\mathbf{k}}$ [see Edwards' eqn. (2.23) and the formula below it]. This result was obtained independently by Novikov (1964) at about the same time, and is usually attributed to him.

A difficulty with the Liouville equation for dissipative systems is that it is impossible to write down its stationary solution in analytical form, as one can write down the Gibbs distribution for thermodynamic equilibrium. However, Edwards realized that an effective substitute is to write down an analytical expression for the distribution over histories, or a *path-integral*. This is his formula (3.7), which provides an exact non-Gaussian distribution over space-time histories of the turbulent velocity. An advantage of this approach is that it allows a calculation also of multi-time statistics, such as the two-time correlation functions, which are of independent interest. From his path-integral formula, Edwards derived a set of statistical field equations (3.5), by a standard method (*e.g.* section 10-1-1 of Itzykson and Zuber 1980). These equations are the main focus of his later analysis. The modernity of Edwards' approach is quite striking: while similar path-integral formulas had been introduced before for linear statistical dynamics by Onsager and Machlup (1953), this may be the first introduction of such a formula for classical nonlinear dynamics. It is a small step to transform Edwards' path-integral expression into the now-standard one for the Martin–Siggia–Rose field-theory with an extra 'response field' (Martin



*et al.* 1973, Janssen 1976, DeDominicis and Peliti 1977). Such path-integrals for stochastically forced Navier-Stokes equations have proved useful in more recent work—*e.g.* the evaluation of tails of probability density functions (PDF's) using instanton methods (Falkovich and Lebedev 1997).

As Edwards notes, his path-integral reformulation of the problem is in principle 'a solution to the problem of turbulence, [but] as it stands it is quite useless in practice.' It can only be useful in conjunction with some method of calculating approximately the integrals of non-Gaussian densities over high-dimensional spaces. To this end, Edwards develops a 'self-consistent' perturbative expansion based upon a physical analogy of turbulence dynamics to stochastic Langevin dynamics. That is, Edwards proposes that the effects of averaging over turbulent fluctuations in his field equations can be subsumed into two dynamical contributions: an 'eddy viscosity' augmenting the damping from molecular viscosity and an effective stochastic force, or 'eddy noise', chaotically generated by the nonlinear dynamics. To develop the necessary formalism for his expansion, Edwards derives in Section 4 of his paper the Gaussian path-integral for a linear Langevin model. His procedure in the following two sections is then to *postulate* such a Gaussian expression as the leading-order approximation to the path-integral for Fourier coefficients of the turbulent velocity. However, this expression contains two unknown functions: an 'eddy-damping coefficient' $R_{\mathbf{k}}$ and an effective 'eddy-noise covariance' $S_{\mathbf{k}}$ acting in each wavenumber $\mathbf{k}$. To determine these functions, Edwards expands his non-Gaussian path-integral density in Hermite polynomials orthogonal with respect to his reference Gaussian (so that the polynomials themselves depend upon the unknowns). For each order, Edwards thereby obtains a set of closed "self-consistent" equations for the functions $R$ and $S$. To first order, they are given by expressions (5.16) and (5.21) in his Section 5, namely,

$$R_{\mathbf{k}} = \int \frac{L_{\mathbf{ljk}} q_{\mathbf{j}}\, d^3\mathbf{j}}{\omega_{\mathbf{k}} + \omega_{\mathbf{j}} + \omega_{\mathbf{l}}}, \quad S_{\mathbf{k}} = \int \frac{L_{\mathbf{kjl}} q_{\mathbf{j}} q_{\mathbf{l}}\, d^3\mathbf{j}}{\omega_{\mathbf{k}} + \omega_{\mathbf{j}} + \omega_{\mathbf{l}}}. \tag{4.3}$$

Here $L_{\mathbf{ljk}}$ are known coefficients, determined from the Navier-Stokes nonlinearity, and $q_{\mathbf{k}}$ is the Fourier transform of the spatial velocity-correlation (or the so-called tensor energy spectrum in $\mathbf{k}$-space). Note from Edwards' equation (5.5), $\omega_{\mathbf{k}} = \nu k^2 + R_{\mathbf{k}}$, that the frequency itself depends upon $R_{\mathbf{k}}$. These sets of equations are the main and general theoretical result of Edwards' paper. Section 7 of the paper gives a similar analysis for multi-time correlation functions.

Technically, there are many points of overlap of Edwards' approach with the somewhat earlier work of Kraichnan (1959, 1961) and Wyld (1961) and the later work of Martin *et al.* (1973), all of whom realized the necessity of two functions like $R_{\mathbf{k}}$ and $S_{\mathbf{k}}$. The first is what Kraichnan calls the 'response function' and the second the 'correlation function' (or, more accurately, their two-particle irreducible 'self-energy' parts). Edwards' self-consistent expansion procedure is formally somewhat different from that employed by Kraichnan, Wyld, and Martin *et al.* As remarked by Martin *et al.*, Edwards' expansion is more akin to the 'quasi-particle' procedures used in condensed matter physics. This is particularly clear in Edwards' use of formal expansions in Hermite polynomials, which are the eigenfunctions of the self-consistent evolution operator [his eqn. (6.16)]. However, Edwards' idea is based on the same physical intuition as that of the other authors. In fact, his Sections 4–6 can be regarded as a precursor to the works of Kraichnan (1970) and Leith (1971), who several years later discovered that Kraichnan's direct interaction approximation (DIA) closure is realized by a self-consistent Langevin model. A number of later simplifications, such as Orszag's (1977) eddy-damped quasi-normal Markovian (EDQNM) closure also have a Langevin realization. Such a model is the essential content of Edwards' Sections 4–6 regarding time-independent statistics, and of Section 7 regarding time-dependent ones.

In Section 8 of the paper Edwards applies his general formalism to a number of concrete problems. In particular, he discusses the approximate eqns. (8.1)–(8.8), for the energy spectrum, namely,

$$\frac{\partial q_{\mathbf{k}}}{\partial t} = -(R_{\mathbf{k}} + \nu k^2) q_{\mathbf{k}} + (S_{\mathbf{k}} + h_{\mathbf{k}}) \tag{4.4}$$

where $R$ and $S$ are given by (4.3). In the text near his eqn. (8.13), Edwards shows that the above equation has the correct thermal equilibrium 'equipartition solution' when the random force has the $k^2$ spectrum required by the fluctuation-dissipation relation. Also, as a consequence of (4.3), he notes in his eqn. (8.6) that

$$\int d^3\mathbf{k}(S_{\mathbf{k}} - R_{\mathbf{k}} q_{\mathbf{k}}) = 0, \tag{4.5}$$

which corresponds to the conservation of energy by the nonlinear terms in Navier-Stokes equations. However, this cancellation is only formal if the integrals diverge. At the top of p. 261, Edwards has an interesting discussion about this 'apparent paradox,' namely that there can still be finite energy dissipation even when the viscosity, $\nu$, vanishes (*e.g.*, Frisch 1995). Edwards points out that the apparent cancellation of integrals in his (8.6) is not 'meaningful' when they are separately divergent. This is very much related to earlier remarks of Onsager (1949) about the impossibility to reorder Fourier series which are not absolutely summable but only conditionally convergent. However, Edwards does not quite reach Onsager's sharp conclusion that energy dissipation is possible without viscosity. Rather, he only concludes that adding a little viscosity makes the integrals convergent and permits their exact cancellation. More recently, Polyakov (1993) has pointed out an analogy of 'inviscid dissipation' in turbulence in two dimensions to conservation-law anomalies in quantum field theory, such as the axial anomaly in quantum electrodynamics.

The case considered by Edwards at some length in his eqns. (8.15)–(8.35) is random forcing with a power-law spectrum. This same problem was later also considered by DeDominicis and Martin (1973) and Yakhot and Orszag (1987)



in any space dimension $d$ (whereas Edwards considered only $d = 3$). The parameter $\alpha$ of Edwards [see his eqn. (8.15)] is the same as the parameter $y$ used by these later authors. Edwards does not consider an expansion in the parameter $\epsilon = 4 + y - d = 1 + y$ (for $d = 3$). However, his self-consistent equations yield similar results as those obtained by the later authors using $\epsilon$-expansion renormalization group methods. His treatment is perhaps more similar to that of Mou and Weichman (1995), who analyzed the same problem using Kraichnan's DIA equations or the mode-coupling approach. In particular, Edwards obtains a power-law energy spectrum through his eqn. (8.30), this result being equivalent to the later authors' result $E(k) \sim k^{-x}$ with

$$x = 5/3 + 2(y - d)/3.$$

(Note there is a typographical error in the first of Edwards' equations, which is missing a minus sign in the exponent.) It is interesting that Edwards explicitly states that the validity of this solution is limited to $-1 < y < 2$, or, equivalently, when $d = 3$, to $0 < \epsilon < 3$. There is an important qualitative change in the problem at $y = 2$ or $\epsilon = 3$, at which the spectrum goes through $E(k) \sim 1/k$, and there begins to be more energy at low $k$ than at high $k$. That is exactly where 'energy cascade' begins. This limitation was also recognized also by DeDominicis and Martin, whereas Yakhot and Orszag imagined that the $\epsilon$-expansion result would be valid all the way to $\epsilon = 4$. In fact, it appears likely, from both physical arguments and work to be described later on a model problem, that intermittency corrections start to appear for $\epsilon > 3$. The restriction of the dimensional analysis results to $0 < \epsilon < 3$ was understood by Edwards, and was probably pointed out in this paper for the first time.

Within the confines of the randomly forced Navier–Stokes equation, the problem of 'true turbulence' corresponds to the case of a compact force in Fourier space, supported at low wavenumbers. Edwards considers this important problem on pp. 261–263 of his paper. In certain limiting situations, he obtains the spectral exponents of Kolmogorov (1941) and Kraichnan (1959). Recognizing that there are potential infrared divergences, Edwards argues that the energy spectral exponent must lie between limits set by the Kolmogorov value of $-5/3$ and the Kraichnan value of $-3/2$. From the point of view of potential intermittency corrections, it is interesting that he only allows spectra that roll off less steeply than Kolmogorov's, whereas we now believe the roll-off to be steeper (*e.g.* Kaneda *et al.* 2002).

Finally, Edwards (1964) makes detailed calculations for two-time correlations of the velocity Fourier amplitudes, on pp. 263–264. He predicts that they will fall off as a Gaussian for small times, as an exponential for intermediate times, and as a power-law for long times. The decay rates that he calculates are all wavenumber dependent. Before this work of Edwards, and the somewhat earlier work of Kraichnan (1959, 1961), the interesting subject of time-correlation functions in turbulence had been neglected. There is no discussion in Edwards of Lagrangian *vs.* Eulerian time correlations, and it is apparent that his predictions must be understood to be for Eulerian time-correlations. However, sweeping effects, which Edwards does not discuss, must come into play in that case, as discussed by Kraichnan (1964) and later by Kraichnan and Chen (1989).

Before concluding this section, we should also point out that Edwards has returned to studies of turbulence several times since his first 1964 paper, even as recently as 2002. We will not make extensive comments on these later forays but make only brief comments on their scope and point out some important subsequent developments.

Following the framework of the 1964 paper, Edwards and McComb (1969) calculated the Kolmogorov constant for the energy spectral density. A notable feature of this paper was the use of a maximum entropy argument to develop a second relationship, in addition to (4.3), between the two functions $R_{\mathbf{k}}$ and $q_{\mathbf{k}}$ in Edwards' self-consistent expansion. This is necessary to fully determine these two functions. The role of entropy and the second law in turbulence remains an intriguing issue. Of course, the maximum entropy principle is unlikely to be strictly valid for a dissipative, far-from-equilibrium system like turbulence. In this respect, the notion of 'relative entropy' discussed somewhat later by Schlögl (1971a,b) may have a better theoretical foundation.

In Edwards and McComb (1971), the authors approximated Kraichnan's response function (see earlier discussion) by using only its inertial range form and obtained a closed-form expression for the spectral density; in particular, they were able to express the viscous cut-off in terms of a Bessel function of the second kind, leading to an exponential fall-off in the far-dissipation range. The ideas of this paper were later extended in Edwards and McComb (1972) to make detailed calculations for a channel flow. In this ambitious work, which aimed to obtain closed-form solutions for the mean velocity, dissipation, and so forth, in a two-dimensional channel flow, the authors were understandably rewarded with only moderate success.

A few years later, Edwards and Taylor (1974) investigated point-vortex models of two-dimensional plasmas and fluids using Gibbsian statistical mechanics of Hamiltonian systems, and discussed cluster formation in terms of 'negative temperatures,' a concept earlier introduced by Onsager (1949). There has been extensive development in this field in the intervening years. After Edwards and Taylor (1974) pointed out the failure of the standard thermodynamic limit for this problem, Lundgren and Pointin (1977) observed that there is a suitable (nonstandard) limit in which $2N$ vortices of strength $\pm 1/N$ are distributed over a flow domain of fixed, finite volume $V$. This permits one to approximate a continuous vorticity distribution of finite energy $E$ by point vortices, in the limit $N \to \infty$. For such a limit the existence of Onsager's negative temperature states has been proved within the microcanonical ensemble, by Eyink and Spohn (1993) and Kiessling and Lebowitz (1997). A number of issues discussed by Edwards and Taylor (1974)—such as the critical energy for appearance of negative temperatures and the possible nonequivalence of canonical and microcanonical ensembles—have now been definitively resolved. The most important development in the field has been the application, independently by



Robert (1990) and Miller (1990), of Gibbsian statistical mechanics directly to the continuum Euler equations without the point-vortex approximation.

In a paper written to commemorate Kubo's sixtieth birthday, Edwards (1980) obtained working expressions for two basic features of turbulence: the single-point probability density function for the velocity fluctuation and the small-scale intermittency. Using polymers with freely hinged chains as the analogy, Edwards closed the problem and showed that the PDF of the single-point velocity has a Gaussian core with stretched exponential tails, which he explicitly obtained. In contrast, experience from simulations and measurement seems to favor slightly sub-Gaussian tails. For the intermittency problem, Edwards used a different analogy to the localized states in disordered semiconductors, and discussed the energy growth of eddies in the inertial range. The jury is still out on these ideas.

Most recently, Edwards and Schwartz (2002) have considered turbulence and surface growth models (particularly focusing, for growth phenomena, on the Kardar-Parisi-Zhang model) and, using path-integral methods, discussed theoretical developments needed for determining two-time correlation functions. This is a direct outgrowth of the first paper, Edwards (1964). A key result is an approximate Markovian equation, *i.e.*, local in time, for the two-time correlation. This approach also yields a natural alternative to the maximum entropy constraint of Edwards and McComb (1969). One of the more characteristic results of the new approach is the stretched-exponential decay of time correlations, rather than a purely exponential form.

It is thus clear that turbulence has gripped the interest of Sam Edwards for many years. His body of work contains many nuggets of technical mastery and intuitive notions, some of which have seen fruition in different ways. The subject has advanced significantly in several directions since Edwards made his entry. In the next two sections, we provide a brief overview of recent developments in the application of field-theoretic methods to the problem of turbulence. Although we will not attempt to trace accurately the influence of Edwards on these recent developments, we shall briefly highlight their relation to his earlier ideas.

## 4.3 The white-noise passive scalar model

As mentioned earlier, the perturbative methods that Edwards helped pioneer have so far not been crowned with absolute success for the problem of Navier–Stokes turbulence. However, there has been recent noteworthy progress on another problem: the advection of a passive scalar by 'synthetic turbulence,' a Gaussian random velocity field which is white-noise in time. This model was introduced by Kraichnan (1968) and, for this reason, is called the *Kraichnan model*. Since an authoritative review of the subject is now available (Falkovich *et al.* 2001), our survey below of the recent work on this model will be brief.

The Kraichnan model considers the concentration $\theta(\mathbf{r}, t)$ of a passive scalar such as a dye or temperature field injected into developed turbulence. The dynamical equation it satisfies is

$$(\partial_t + \mathbf{U}\cdot\boldsymbol{\nabla})\theta = \kappa\boldsymbol{\nabla}^2\theta + F. \tag{4.6}$$

Here, $F(\mathbf{r},t)$ is a Gaussian white-noise source term of a passive scalar, similar to that considered by Edwards for the velocity, with zero mean and covariance $F(\mathbf{r}-\mathbf{r}')\delta(t-t')$. The advecting incompressible velocity field $\mathbf{U}(\mathbf{r},t)$ is also a Gaussian random field with zero mean and covariance

$$\langle U_i(\mathbf{r},t)U_j(\mathbf{r}',t')\rangle = D_{ij}(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{4.7}$$

The cases of greatest interest are those for which the source $F$ is supported at low-wavenumbers, while the velocity field $\mathbf{U}$ has a power-law spectrum $\sim k^{-(1+\xi)}$ at high wavenumbers, for any $0<\xi<2$. This corresponds to a scaling

$$d_{ij}(\mathbf{r}) := D_{ij}(\mathbf{0}) - D_{ij}(\mathbf{r}) \sim \frac{D_1}{d-1}r^{\xi}\left((d-1+\xi)\delta_{ij} - \xi\frac{r_i r_j}{r^2}\right) \tag{4.8}$$

for $r\to 0$ in space dimension $d$ [see eqn. (48) of Falkovich *et al.* (2001)].

As was first realized by Kraichnan himself (see Kraichnan 1968), the special feature of the model is that there is no closure problem, independently of the precise form of the function $d_{ij}(\mathbf{r})$. The single-time scalar correlation functions $C_n(\mathbf{r}_1,\ldots,\mathbf{r}_n;t) = \langle\theta(\mathbf{r}_1,t)\cdots\theta(\mathbf{r}_n,t)\rangle$ satisfy the equation

$$\begin{aligned}\partial_t C_n(\mathbf{r}_1,\ldots,\mathbf{r}_n;t) = &-\sum_{1\le l<m\le n}[d_{ij}(\mathbf{r}_l-\mathbf{r}_m)+2\kappa\delta_{ij}]\nabla^i_{\mathbf{r}_l}\nabla^j_{\mathbf{r}_m}C_n(\mathbf{r}_1,\ldots,\mathbf{r}_n;t)\\ &+\sum_{1\le l<m\le n}F(\mathbf{r}_l-\mathbf{r}_m)C_{n-2}(\mathbf{r}_1,\ldots,\widehat{\mathbf{r}_l},\ldots,\widehat{\mathbf{r}_m},\ldots,\mathbf{r}_n;t).\end{aligned} \tag{4.9}$$

Here the summation is over all pairs $l<m$ of integers $l,m=1,\ldots,n$ while the hats $\widehat{\ }$ over the position vectors indicate their omission from the correlation function. Since this equation for $C_n$ involves only the lower-order correlation function $C_{n-2}$, it is possible, in principle, to mathematically solve this hierarchy of equations inductively for all of orders of scalar correlations.

An important problem, as in the case of critical phenomena, is the prediction of the scaling exponents for the small-scale scalar field. The phenomenology in this case is quite similar to that for the turbulent velocity itself. One imagines that there is a cascade of the scalar 'energy' or intensity, $\frac{1}{2}\int d^d\mathbf{r}\,\theta^2(\mathbf{r},t)$, from the injection scale $L$ to small scales. When the scalar is weakly diffusive, there is a large range of scales, the so-called inertial-convective range, over which the flux of scalar intensity is constant. However, this is only true in an average sense, and fluctuations in individual realizations of the cascade do develop. These fluctuations become large as one considers increasingly smaller scales. This property of 'intermittency' is the origin of the *anomalous scaling* for the scalar structure functions defined by

$$\langle[\theta(\mathbf{r},t)-\theta(\mathbf{0},t)]^p\rangle \sim \vartheta^p\left(\frac{r}{L}\right)^{\zeta_p} \tag{4.10}$$



where $\vartheta^2 = \langle\theta^2\rangle$ is the mean-square scalar fluctuation and $\zeta_p$ is a scaling exponent. Classical dimensional analysis of Obukhov (1949) and Corrsin (1950), itself an outgrowth of Kolmogorov (1941), predicts that $\zeta_p = p/3$. In fact, it is known from experiments on real turbulent scalars that $\zeta_p$ is a concave, nonlinear function of the order $p$ of the structure function (Antonia *et al.* 1985, Meneveau *et al.* 1991, Chen and Kao 1997, Moisy *et al.* 2001, Skrbek *et al.* 2002). This type of anomalous scaling law is now called *multifractal*, after an intuitive interpretation by Parisi and Frisch (1985); see also Mandelbrot (1974). Multifractality involves an infinite concave family of exponents and is fundamentally different from the scaling observed in classical critical phenomena.[2]

Returning to the Kraichnan model, the velocity field in the model does not possess anomalous scaling, since it is Gaussian and unifractal. Surprisingly, however, the scalar field advected by this Gaussian velocity field *does* show multifractal scaling. Although the input velocity is self-similar, it induces a scalar cascade through which intermittency develops in successive steps from large scales to small. In fact, it is possible to guess from Edwards' work the spectral exponent $\xi$ of the velocity [see eqn. (8)] at which multifractality begins to develop. As was noted by Edwards for the Navier–Stokes fluid stirred by a random force with power-law spectrum, the energy spectrum has the form $k^{1-2\epsilon/3}$ for $0<\epsilon<3$. In this range, there is more energy at high wavenumbers than at low wavenumbers, and no energy cascade occurs. Precisely at $\epsilon=3$ the velocity spectrum makes the transition from most energy at high wavenumbers to most at low wavenumbers. Edwards realized that dimensional reasoning breaks down for $\epsilon>3$ and that possible corrections to the scaling laws can occur there. Since the energy spectrum of the velocity field in the Kraichnan model has the power-law form $\sim k^{-(1+\xi)}$, there is the formal identity $\xi = 2(\epsilon-3)/3$. Hence, it is likely that anomalous scaling begins in the Kraichnan model precisely at $\xi=0$.

In analogy with $\epsilon$-expansions used in critical phenomenon, it is suggested that the natural perturbation parameter is not $\epsilon$, as considered by DeDominicis and Martin (1979), and Yakhot and Orszag (1986), but instead $\epsilon-3$, or $\xi$. The first to realize this fact for the Kraichnan model were Gawędzki and Kupiainen (1995), who carried out the corresponding expansion. Fortunately, the Kraichnan model is exactly solvable and the scalar statistics are Gaussian for $\xi=0$. Indeed, setting $\xi=0$ in (4.8) and substituting into (4.9) gives

$$\begin{aligned}\partial_t C_n(\mathbf{r}_1,\ldots,\mathbf{r}_n;t) = &-\sum_{1\le l<m\le n}(D_1+2\kappa)\delta_{ij}\boldsymbol{\nabla}^i_{\mathbf{r}_l}\boldsymbol{\nabla}^j_{\mathbf{r}_m}C_n(\mathbf{r}_1,\ldots,\mathbf{r}_n;t)\\ &+\sum_{1\le l<m\le n}F(\mathbf{r}_l-\mathbf{r}_m)C_{n-2}(\mathbf{r}_1,\ldots,\widehat{\mathbf{r}_l},\ldots,\widehat{\mathbf{r}_m},\ldots,\mathbf{r}_n;t).\end{aligned} \tag{4.11}$$

[2] Our point is not that multifractality *cannot* appear in ordinary critical systems, but that it is different from the 'classical critical scaling' in which only a finite number of exponents are readily apparent. It is, in fact, possible (see, *e.g.*, Fourcade and Tremblay 1995) for more 'exotic' operators than those considered in 'classical' critical phenomena to possess multifractal properties.

The very rough velocity field for $\xi = 0$ acting at high wavenumbers is seen to mimic exactly a molecular diffusion. This equation for translation-invariant correlation functions is just a multi-dimensional heat equation, and it is not hard to show that Gaussian correlation functions, satisfying Wick's theorem, are the solution.

From this point Gawędzki and Kupiainen were able to develop an expansion for the anomalous exponents of the scalar (see also Gawędzki and Kupiainen 1995; Bernard *et al.* 1996), with the result that

$$\zeta_n = \frac{n(2-\xi)}{2} - \frac{n(n-2)}{2(d+2)}\xi + O(\xi^2).$$

At nearly the same time it was realized by Chertkov *et al.* (1995) and Chertkov and Falkovich (1996) that the scalar statistics also become Gaussian for $d = \infty$. They worked out a corresponding expansion in $1/d$, yielding a result consistent with the above.[3] The Kraichnan model also simplifies for a smooth velocity at $\xi = 2$, the so-called Batchelor regime of the passive scalar. In that case, all $\zeta_n = 0$, in the sense that the correlation functions are logarithmic, instead of being power-laws. In this limit, an expansion was worked out by Shraiman and Siggia (1995, 1997). This is technically a more difficult limit than the other two: it is a singular perturbation problem with a boundary-layer, for which the relevant expansion parameter turns out to be $(2-\xi)^{1/2}$. The problem has thus been worked out only for the triple correlation $n = 3$. All of these authors realized that, in the Kraichnan model, so-called 'zero modes,' that is, stationary solutions of the homogeneous version of equation (4.9), play the key role in the development of anomalous scaling. This is due to the fact that equations formulated for the scalar correlation functions are linear with known (singular diffusion) operators. This linearity property is essentially unrelated to the linearity of the advection-diffusion equation. However, the fact of closure, *i.e.*, the existence of closed linear equations, does depend upon the linearity of advection-diffusion and also the white-noise character of the velocity; as already implied, it is this combination that truncates the hierarchy in the Kraichnan model.

All these perturbative results have now been checked by clever Lagrangian numerical methods, for the $\xi$ and $(2-\xi)^{1/2}$ expansion by Frisch *et al.* (1998, 1999), and Gat *et al.* (1998), and for the $1/d$ expansion by Mazzino and Muratore-Ginanneschi (2001). To know the scalar field at position $\mathbf{x}$ and time $t$, it is enough to track the corresponding tracer particle back to its (Lagrangian) initial position. It follows that the evolution operator for the $n$-point average of the scalar coincides with that for the probability that $n$ tracer particles from given positions reach new positions $\mathbf{x}_k$, $k = 1, \ldots, n$ after time $t$. For the $n$-point function, it is only necessary to focus on the relative evolution of $n$ particles simultaneously. This reduces the problem to a study of the evolution of the geometry

[3] The $1/d$ expansion for equilibrium lattice systems was introduced by Fisher and Gaunt (1964).



of polyhedra with $n$ vertices. Deviations from dimensional estimates (*i.e.* anomalies) are traceable to the nontrivial evolution of these geometric objects as advected by the flow. Among all the geometric figures that grow in time according to dimensional estimates, the ones that matter are those whose shapes are preserved: these statistically conserved objects are the ones that dominate the behaviour of scales in the inertial range and the anomaly of the exponents. This study also reveals the role of 'hidden' statistical integrals of motion. Indeed, the 'zero modes' are statistical integrals of motion of the Lagrangian fluid particles.

The perturbation theories originally worked out for the Kraichnan model did not use a Martin–Siggia–Rose field-theory formulation of the sort that Edwards helped to pioneer. Nor, for that matter, did they use the renormalization group (RG) as a basis to organize the expansions. However, in later works, this has been done, first by Gawędzki (1997) and later in an extensive series of work by the St. Petersburg school in Russia (*e.g.* see Adzhemyan *et al.* 2001, 2002). As might be expected, RG is a more important tool when working out higher terms of the expansions: with its aid the $\xi$-expansion has now been carried out to third-order (Adzhemyan *et al.* 2001). Even more interestingly, the same perturbative RG approach has been extended to a generalized model, still with a Gaussian random advecting velocity field, but now with correlations exponentially decaying in time rather than delta-correlated (Adzhemyan *et al.* 2002). This is a step in the direction of greater realism of the models. Some previous conjectures for 'additive' operator product expansions as the basis of turbulent multifractality (Eyink 1993; Lebedev and L'vov 1994) have been confirmed in the models by using perturbative RG techniques.

Thus, for a nontrivial model problem and some generalizations, the state of our understanding of anomalous scaling is now essentially as good as for three-dimensional critical phenomena, where there are Borel-resummed $\epsilon$-expansion results, high-temperature series expansions, and numerical Monte Carlo results, which agree to several decimal places. In both cases, rigorous mathematical proofs are still lacking, but it is quite clear that scaling exponents exist and what their numerical values are. Needless to say, we have far from adequately reviewed all the results that have been obtained by now for the Kraichnan model of a passive scalar. For example, new insight has been obtained into dissipative anomalies in turbulence related to the non-uniqueness and stochasticity of Lagrangian particle paths for a non-differentiable advecting velocity. White-noise advection models have also been fruitfully investigated for related problems such as compressible turbulence, magnetohydrodynamics and the coil-stretch transition of polymers in turbulence. We refer to Falkovich *et al.* (2001) for a fuller discussion of these many aspects.

## 4.4 Navier–Stokes turbulence

The results we have described for the Kraichnan model have already had a significant impact on our understanding of Navier–Stokes turbulence. There is now rather general agreement that velocity structure functions will show anomalous, multifractal scaling. There was already good empirical evidence for this, both experimental (Anselmet *et al.* 1985; Sreenivasan and Dhruva 1998) and numerical (Cao *et al.* 1996). It is hard to imagine that anomalous scaling would arise in the cascade for the passive scalar and not for the nonlinearly self-interacting velocity field. In this respect, the Kraichnan model results have served much the same purpose as Onsager's exact solution of the two-dimensional Ising model, which eventually convinced most statistical physicists that Landau's mean field theory needed to be replaced. In the same way, the fact that Kolmogorov-style dimensional analysis fails in the Kraichnan model lessens one's faith that these same arguments will succeed when applied to Navier–Stokes turbulence. A better theory is clearly needed.

Technically, it has not proved possible to carry over readily the formal methods successfully employed in the Kraichnan model to Navier–Stokes equations. This, also, is very much as for the two-dimensional Ising model, whose specific methods of solution (such as infinite-dimensional Lie algebras, spinors, and Toeplitz determinants) played no direct role in the apparatus of the successful general methods such as the renormalization group and the $\epsilon$-expansion. The only method used in Onsager's solution that has subsequently shown general applicability to a wide class of systems is the transfer matrix. Likewise, it may be that there are certain features of the solution of the Kraichnan model that are more general and can be carried over to Navier–Stokes turbulence and to other similar nonequilibrium scaling problems, such as random surface growth. The concept of a 'zero mode' seems a plausible candidate. Zero modes can also be considered in nonlinear dynamics (*e.g.* in shell models or Navier–Stokes turbulence) by working with the linear Hopf equation or linear Liouville equation or, equivalently, with the infinite linear hierarchy of equations for multipoint correlation functions. The linearity of the scalar advection-diffusion equation is not a prerequisite for the existence of zero modes, and it is likely that they play an important role in anomalous scaling more generally.

The perturbative expansion techniques employed successfully in the Kraichnan model have not so far found any success in Navier–Stokes turbulence. It is quite likely, as we discussed earlier, that randomly stirred fluids with power-law forcing spectrum first develop intermittency at $\epsilon = 3$. However, the statistics do not become Gaussian in that limit and there is no other obvious analytical simplification at $\epsilon = 3$. So, an expansion in $\epsilon - 3$ does not look very feasible. It is also a frequent speculation that Navier–Stokes turbulence should simplify in infinite-dimensional space (Frisch and Fournier 1978), but this has not yet been demonstrated. Yakhot (2001) has proposed an expansion in $d - d_c$, with $2 < d_c < 3$ a critical dimension where the energy cascade changes from inverse to direct, and made some progress by attributing a simple behaviour for pressure terms. One likely hope for a successful perturbative treatment is the $1/N$ expansion for Kraichnan's 'random coupling' model (Kraichnan 1961), in which $N$ copies of the Navier–Stokes equation are coupled together with quenched random parameters. It is known for this and related models (Kraichnan 1961; Eyink 1994a; Mou and Weichman 1995) that Kraichnan's DIA closure becomes exact and the statistics become Gaussian at $N = \infty$. Furthermore, in an $N$-component version of a shell model it has been shown numerically that the anomalous scaling corrections to Kolmogorov's (1941) arguments vanish proportionally to $1/N$ (Pierotti 1997). Thus, the anomalous exponents should be perturbatively accessible in the shell model by a $1/N$ expansion. Unfortunately, for Navier–Stokes turbulence, the DIA closure is not consistent with Kolmogorov (1941) scaling and a Lagrangian formulation of the random-coupling model would need to be devised (Kraichnan 1964). None of these steps appears to be trivial.



## 4.5 Conclusion

The turbulence problem remains a hard nut to crack. Anomalous scaling almost certainly occurs, but no controlled approximation to the exponents exists. Furthermore, there are many other flow properties in turbulence that we would like to calculate, not just scaling exponents. More practical, and also physically very interesting, are quantities such as drag coefficients, mixing efficiency, spread rates, single-point probability density functions, and mean velocity profiles. So far, none of these aspects of the problem can be reliably calculated at high Reynolds numbers.

We do not, however, wish to leave the impression that the situation is hopeless. Despite the lack of adequate analytical tools, quite a lot of understanding has been gained, through the work of many people, Sam Edwards among them. Much of the lore is reasonably well established on the basis of experiment, simulation, and theoretical arguments. As we have argued above, multifractal scaling of the turbulent velocity field is now doubted by few, although occasional papers still appear claiming the contrary. It is clear in flow experiments (Sreenivasan and Dhruva 1998) and simulations (Cao *et al.* 1996), and has been established theoretically for the Kraichnan model. Likewise, there is good evidence for a dissipative anomaly in the conservation of energy at zero viscosity. It is observed in experiments (Sreenivasan 1984) and simulations (Kaneda *et al.* 2003), and there are known theoretical mechanisms to produce it (Onsager 1949; Eyink 1994b). Concepts such as fractality of isosurfaces (Sreenivasan 1991), fusion rules for powers of velocity-gradients (Fairhall *et al.* 1997), Kolmogorov's refined similarity hypothesis connecting scaling of velocity-increments and dissipation (Stolovitzky and Sreenivasan 1994), stochasticity of Lagrangian particle trajectories (Bernard *et al.* 1998; Yeung 2002), and many other key ideas, seem well-founded and likely to survive into the future. Our ability to calculate with the Navier–Stokes equations continues to extend to higher Reynolds numbers because of advances in computing power, and new experimental techniques (Donnelly and Sreenivasan 1998) and modeling techniques (Meneveau and Katz 2000) extend that capacity.

The present status for the 'turbulence problem' seems to us really not so different from that for other strong-coupling problems in field theory, *e.g.*, the color confinement problem in QCD. In the confinement problem there are also heuristic ideas [the QCD vacuum is a color magnetic monopole condensate, a Type II chromomagnetic superconductor with confined chromoelectric flux tubes acting as 'strings' between color charges; see Mandelstam (1978), and 't Hooft (1978, 1981)], but there is great difficulty in making quantitative calculations. Direct numerical simulations, *i.e.*, lattice QCD (Wilson 1974) allow us to calculate, in principle, anything we wish (*e.g.* the hadron spectrum), but in practice computer limitations confine us to modest results for the foreseeable future. Nevertheless, many aspects of QCD are regarded as reasonably well-established (*e.g.* a mass gap, color confinement, chiral symmetry-breaking), even though they are not rigorously proved. Turbulence is very similar. It is no accident that both these problems ended up as Clay Institute Millenium Prize Problems: proving the mass gap for four-dimensional quantum non-abelian gauge theory, and proving regularity of Navier–Stokes solutions at high Reynolds numbers. They are both strong-coupling problems and those *are* mathematically hard. However, in both cases we have a lot of very good insights and ideas that can spur intuitively new developments. The difficulties that remain should not lead us to despair, but, instead, ought to inspire us to greater imagination and ingenuity—qualities that Edwards has always had in generous quantities.

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It is a pleasure to be a part of the celebration honoring Sir Sam Edwards. In this chapter, we have attempted to outline his contributions to the important topic of turbulence. It is not likely that we have done full justice to Sam's effulgent spirit, which the subject clearly did not contain entirely. Asked how one would know if one has arrived at the right answer to the problem, KRS well remembers Sam saying, 'A little angel will whisper in your ears that you are right.'

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