Statistical Dependence of Inertial Range Properties on Large Scales in a High-Reynolds-Number Shear Flow

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For turbulent velocity data measured in high-Reynolds-number atmospheric surface layer, we show that the observed dependence of the inertial range statistics on large scale fluctuations is a manifestation of the mixed averaging of regions of different local Reynolds number. This dependence can be subsumed by the Obukhov conditional averaging and the second refined similarity hypothesis of Kolmogorov. We identify a subset of velocity increments which seem to possess no large-scale dependence. [S0031-9007(96)00767-3]

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Fully turbulent flows at high Reynolds numbers consist of a wide range of scales which are classified, somewhat loosely, into "large" and "small" scales. The small scales include the dissipative range responsible for most of the energy dissipation and the intermediate inertial-range scales that bridge the large scales, on the one hand, and dissipative scales, on the other hand. The phenomenology of turbulence rests principally on the premise that, at high Reynolds numbers, the small scales are statistically independent of the large scales. While there is ample evidence that this is approximately so, it has been appreciated that the picture is not correct in detail [1–4] and that the small-scale properties depend to some degree on the large scale. Despite much work, one does not understand the precise nature of this dependence. The intent of the Letter is to improve this understanding by examining conditional statistics of inertial scales in the framework of Obukhov's pure ensemble [5] and Kolmogorov's refined similarity hypotheses [6].

Specifically, we were motivated by the measurements of Praskovsky *et al.* [7] which showed that the inertial-range scales are statistically correlated with the large scale. Consider the so-called longitudinal velocity increments $\Delta u_r \equiv u(x + r) - u(x)$, where u and r, respectively, are the velocity component and the separation distance in the direction x. If r is in the inertial range, the moments of Δu_r are presumed to be purely inertial-range properties, and so one would expect them to be statistically independent of the large-scale velocity. By conditioning the moments of Δu_r on various values of (a suitable choice of) the large-scale velocity, it was shown in Ref. [7] that statistical independence holds at best for small magnitudes of the large-scale velocity; for magnitudes in excess of, say, a standard deviation of its variation, the moments of Δu_r showed a strong dependence on the large-scale velocity. This coupling between the large scale and the inertial scales could come about by the explicit appearance of the large-scale velocity as a scaling factor [8,9], but it is also likely that it may come about because of the intermittency of the energy dissipation rate, as envisaged in the second refined similarity hypothesis (RSH) of Kolmogorov [6].

For present purposes, RSH can be stated as follows: *Provided the local Reynolds number is high enough,* the quantity that is universal in the inertial range is *not* the velocity increment Δu_r but the nondimensional variable V given by

$$V = \Delta u_r/(r\varepsilon_r)^{1/3}, \tag{1}$$

where

$$\varepsilon_r = \frac{1}{r}\int_x^{x+r} \varepsilon(x')\,dx'. \tag{2}$$

Here, $\varepsilon(x)$ is the energy dissipation rate at the position x, and the integral in Eq. (2) is taken over the distance r; for r much smaller than the large scale L, ε_r is the locally averaged dissipation rate. The local Reynolds number is defined as $\mathrm{Re}_r = r(r\varepsilon_r)^{1/3}/\nu$, ν being the kinematic viscosity coefficient. Obukhov [5] presumed that Kolmogorov's [10] original theory holds exactly in the so-called "pure regimes," or regions of the flow where the energy dissipation assumes some fixed value. The averages over regions with variable ε can be termed, following Obukhov, as "mixed averages." Because of the strong intermittency of the energy dissipation, the local Reynolds number will vary from one pure regime to another. Thus, mixed averages correspond to regions with varied values of the local Reynolds number.

We shall now consider tests that shed some light on the issue of whether the observed statistical dependence is subsumed by the RSH and Obukhov's mixed averages. We do this by analyzing the turbulent velocity data in the atmospheric surface layer measured at a few meters height above the ground, at a Taylor microscale Reynolds number of about 2000. These data have been analyzed for various purposes in Refs. [11,12], where more details can be found. The data consisted of a contiguous record of approximately 2.5×10^6 η, where $\eta = (\nu^3/\langle\varepsilon\rangle)^{1/4}$ is the Kolmogorov length scale, obtained under steady wind conditions with a mean velocity of 6.7 ms^{-1}. The inertial

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range, as judged from the linear behavior of the third-order structure function [13], can be said to extend from about 40η to about 800η. (The scaling range from the power spectrum is more extensive). We have used Taylor's frozen flow hypothesis, and replaced the energy dissipation by the component $(\partial u/\partial x)^2$. Even though the nature of these approximations is not clear, there are indications [14] that they are reasonably benign for present purposes. With one exception, all the following figures have been presented for fixed inertial-range separation distances $r = 560\eta$, but we have verified that the results are essentially the same for half that separation distance.

Before proceeding further, it is necessary to choose a quantity that serves as an appropriate surrogate of the large-scale velocity, u_L. The first consideration in this respect is that the very largest scale velocity fluctuations (containing, say, about 20% of the kinetic energy) are flow dependent. However, if we filter out those largest scales we free ourselves from flow dependent effects. The results to be presented below use velocity data in which the signal below 1 Hz has been filtered out; we treat this filtered signal as our velocity $u(x)$. The effects of this filtering will be discussed below briefly. Two natural choices for the large scale (with the caveat mentioned above) are $u_L = u(x)$ and $u_L = u(x + r/2)$. The first choice is suggested by Kolmogorov's definition of local homogeneity [10] (see also [1], Sec. 21.2), according to which, for a fixed value of $u(x_0, t_0) = u^*$, the probability density function of $u(x_0 + r, t_0) - u^*$ is dependent of t_0, x_0, and u^*. The second choice, namely, $u_L = u(x + r/2)$, is appealing because of the symmetry with respect to the interval $(x, x + r)$, as well as for a deeper reason. To see this latter, consider the equivalent effect for a stationary Gaussian random function $g(t)$ (assumed without loss of generality to have zero mean). It is well known that the statistics of such functions are completely determined by the correlation function $\rho(\tau) = \langle g(t + \tau)g(t)\rangle/\sigma^2$, where $\sigma^2 = \langle g^2\rangle$. For this class of random functions, it can be shown that

$$\begin{aligned}\langle [g(t + \tau) - g(t)]^2 | g(t + \tau_0)\rangle \\ &= \sigma^2\{2[1 - \rho(\tau)] - [\rho(\tau - \tau_0) - \rho(\tau_0)]^2\} \\ &\quad + [\rho(\tau - \tau_0) - \rho(\tau_0)]^2 g(t + \tau_0)^2. \qquad (3)\end{aligned}$$

Equation (3) shows that there is a strong statistical dependence between $g(t + \tau) - g(t)$ and $g(t + \tau_0)$ when $\tau_0 = 0$, but that this dependence vanishes when $\tau_0 = \tau/2$. This result, which can be called "kinematical," suggests that the interaction between small and large scales is best studied by conditioning on the velocity at the center of the interval, rather than on that at its edges.

The continuous curve in Fig. 1 shows the conditional variance of Δu_r for fixed u_0 as a function of u_0, where $u_0 \equiv u/u'$; $u = u(x + r/2)$ is taken as the representation large-scale velocity u_L. The curve is roughly parabolic, and only when $|u_0|$ is small may one consider Δu_r to be roughly independent of u. This result is in agreement with that of Ref. [7]. We note that a different definition of u_0 as the average of $u(x)$ and $u(x + r)$, or as the integrated average velocity over the interval $(x, x + r)$, would not alter this behavior. We also note that the unfiltered data yield essentially the same results except that filtering restores greater symmetry to the curves of Fig. 1. Further, the removal of the very largest scales makes the results of Fig. 1 both qualitatively and quantitatively similar for a variety of flows (which otherwise show some differences). The figure suggests that local sweeping, unless it is small, makes inertial-range quantities depend on large-scales. As we showed above, this dependence would vanish for Gaussian fields.

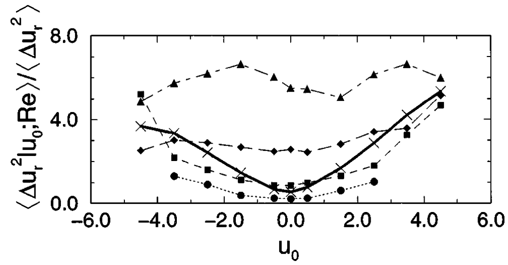

FIG. 1. The conditional expectation of Δu_r^2 for a given u_0, plotted as a function of u_0. $u_0 = u/u'$, where u is the turbulent velocity fluctuation (see text) at the midpoint of the interval $(x, x + r)$. Crosses and the continuous line correspond to the average over the entire domain; circles to the range $0 \le \mathrm{Re}_r < 2800$; squares to $2800 \le \mathrm{Re}_r < 5600$; rhomboids to $5600 \le \mathrm{Re}_r < 9300$; triangles to $\mathrm{Re}_r \ge 9300$.

We have so far not paid attention to whether or not the local Reynolds number is high. Figure 1 also shows the conditional variance of Δu_r for several ranges of local Reynolds numbers. Narrower windows on the Reynolds number—which is to say, narrower windows on ε_r—would have been desirable, but convergence difficulties due to finite data record prevented this refinement. The windows are chosen such that convergence errors do not mask data trends. The results, shown in Fig. 1, lead to two conclusions. First, when the local Reynolds number is large, say, greater than about 5600, the dependence on u_0 of the conditional variance vanishes entirely. Thus, the dependence apparent from the continuous curve seems to be the result of ignoring the requirement that the local Reynolds be sufficiently high. A second conclusion from the lower Reynolds number ranges—namely, $\mathrm{Re} \le 2800$, and $2800 \le \mathrm{Re} \le 5600$—is that, even for these cases, there is less dependence on u_0 than for the continuous curve. This may seem counterintuitive because these two local Reynolds numbers are lower than that for the entire data. Note, however, that each data set corresponds, in a rough sense, to Obukhov's pure regimes, whereas averaging over the entire data corresponds to the mixed regime. Therefore, the data suggest that statistical independence holds better in pure regimes (as approximated by the restriction to a band of local Reynolds numbers) than in

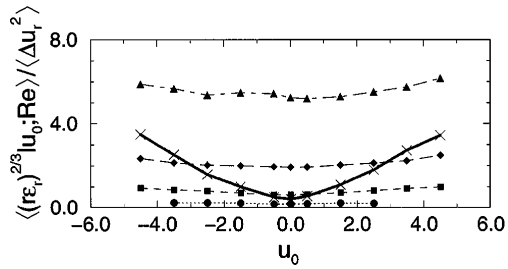

FIG. 2. The conditional expectation of $(r\varepsilon_r)^{2/3}$. Symbols as in Fig. 1.

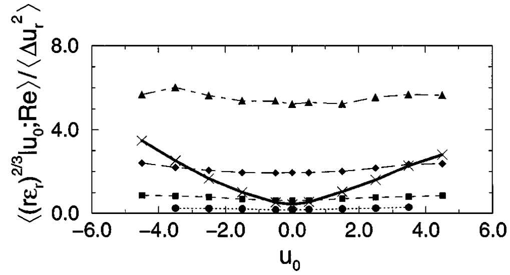

FIG. 4. Same quantities as in Fig. 2, except that u is now defined as the velocity at the position x. The resulting graphs are very similar to those of Fig. 2.

mixed ones. An explanation for this can again be that, for inertial range scales, the statistics of Δu_r for fixed ε_r is nearly Gaussian [11,12]. The dependence observed at small Re_r can probably be ascribed to the fact that the conditioning windows on Re_r are a bit too wide. While this width does not affect the high Re_r regimes, the dependence of V on Re_r for smaller values of Re_r makes a wide window effectively a mixed regime [15]. The conjecture, then, is that the finer the conditioning, the less the large-scale dependence in each pure regime, even if Re_r is not especially large.

Figure 2 shows similar conditional expectations of $(r\varepsilon_r)^{2/3}$ for the same inertial-range distance r as in Fig. 1. The continuous curve for the mixed case is rather similar to the mixed case of Fig. 1. It is also apparent, perhaps with greater clarity than in Fig. 1, that the dependence vanishes nearly entirely when the averaging is restricted to pure regimes.

Since the variances of both Δu_r and $(r\varepsilon_r)^{1/3}$ in the mixed regimes show similar dependence on u_0, it might be surmised that the ratio $V = \Delta u_r/(r\varepsilon_r)^{1/3}$ would show substantially smaller dependence on u_0. Recalling [11,12] that the quantity V for fixed Re_r and inertial range r is nearly Gaussian (albeit skewed), one might expect that this near Gaussianity will restore the large-scale independence to the inertial-range variable V. This is to be expected because the ensemble of pure regimes coincides [15] with the ensemble of an appropriate fractional Brownian motion, which is a Gaussian process. Figure 3 bears out this expectation. The significant aspect of this figure is that the variance of the variable V is independent of u_0 not only for high local Reynolds numbers, but also for the mixed case. Interestingly, in pure regimes where the local Reynolds number is less than about 5600, the conditional expectation shows non-negligible dependence on the large scale. As explained above, we attribute this to the finite-sized windows on the conditioning variable. The picture is entirely consistent with RSH.

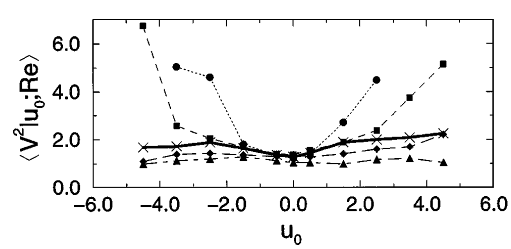

FIG. 3. The conditional expectation of V^2. Symbols as in Fig. 1.

From considerations already discussed, we may expect that the results would be different if the large-scale velocity were identified with the velocity at the end of the interval $(x, x + r)$. The new conditional expectation of $(r\varepsilon)^{2/3}$ (Fig. 4) shows no differences from Fig. 2 and needs no further comment. Figures 5 and 6 correspond to Δu_r and V, respectively. The common feature of Figs. 2 and 5 is the statistical dependence for the mixed case—which, as already mentioned, seems to be a robust result. It is, however, seen from Fig. 5 that Δu_r possesses nontrivial statistical dependence on the large scale no matter what the local Reynolds number (except, perhaps, for the lowest Reynolds number cases). This striking departure from Fig. 1 can be understood by recalling that an equivalent dependence persists even for a Gaussian process, indicating that this effect is mainly kinematical in origin. This dependence is expected in general [16], as can be seen by expanding the conditional expectation Δu_r^2 in terms of u_0,

$$\langle [u(x + r) - u(x)]^2 | u(x)\rangle = [\langle u(x + r)^2 | u(x)\rangle - 2\langle u(x + r) | u(x)\rangle u(x) + u(x)^2] \approx u'^2 + u(x)^2 \,, \tag{4}$$

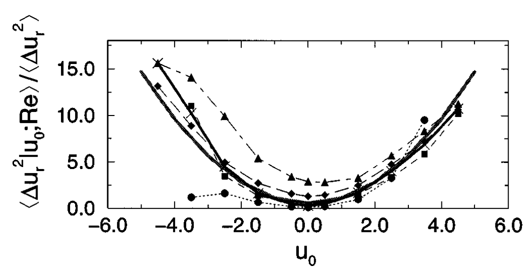

FIG. 5. Same quantities as in Fig. 1, with u as the velocity at position x. Except for the lowest Reynolds number range for which statistical dependence seems to vanish for $u_0 < 0$, the conditional variance depends strongly on u_0 for all cases. The thick grey line corresponds to Eq. (4), after appropriate normalization.

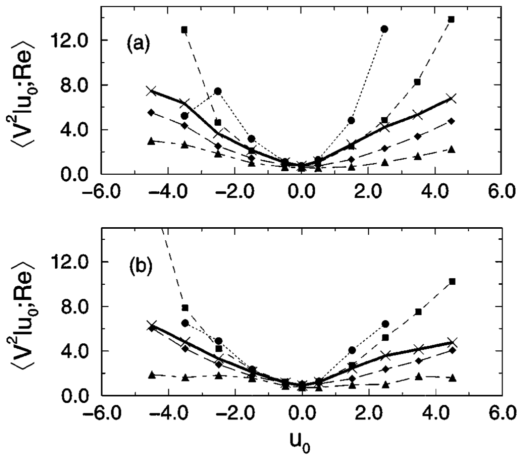

FIG. 6. Same quantities as Fig. 3, with u as the velocity at x. (a) Corresponds to $r/\eta = 560$, and (b) to $r/\eta = 280$. For the mixed regime, the variable V shows a weaker dependence on u_0 than does Δu_r. This dependence diminishes with increasing Reynolds number, and is especially negligible for the range $\mathrm{Re}_r \geq 9300$, $r/\eta = 280$.

where the assumption of a weak correlation between $u(x + r)$ and $u(x)$ was used in the second approximate equality. The parabolic dependence with respect to u_0, expected from Eq. (4), corresponds to the thick grey line in Fig. 5. Thus, a flat curve in Fig. 1 may well transform itself in principle to the parabola of Fig. 5. Similarly, Fig. 6(a) shows that V in the mixed regime has a weaker (though non-negligible) dependence, but becomes weaker still if Re_r is high enough. This is particularly so in Fig. 6(b) for $r = 280\eta$. In any case, the persistence of the dependence of V on the large scale is apparent, but again, its origin is kinematical.

In short, the statistical dependence vanishes in pure ensembles and the variable V is close to being statistically independent of the large scales—at least when the local Reynolds number is sufficiently high. This is in agreement with RSH. The need for large local Reynolds number is not surprising: for, otherwise, the scale separation between the large and small scales becomes narrow.

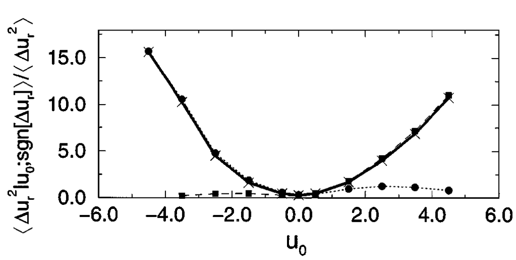

FIG. 7. The conditional variances of Δu_r (crosses and full line), Δu_r^+ (circles and dotted line), and Δu_r^- (squares and dashed line) for a given u_0, plotted as a function of u_0, with u taken as the velocity at x. Note that the conditional variance of Δu_r^+ is independent of u_0 when the latter is positive, and that of Δu_r^- is similarly independent of u_0 when the latter is negative.

Other interesting observations can also be made. For example, it was suggested in Ref. [17] that one can get better insight into the scaling properties of velocity increments by considering their positive and negative parts separately. We consider this aspect briefly in Fig. 7, where the variances of $\Delta u_r^+ \equiv \Delta u_r \geq 0$ and $\Delta u_r^- \equiv \Delta u_r < 0$ are plotted against u_0 as before. For Δu_r^+, the dependence is unchanged from that of full Δu_r when $u_0 < 0$, but vanishes entirely when $u_0 \geq 0$. Just the opposite is true for Δu_r^-. Whether this result is kinematic or dynamical, the observation suggests that the variable ξ defined as

$$\xi = \Delta u_r^+ \text{ when } u \geq 0 \text{ and } \Delta u_r^- \text{ when } u < 0$$

is likely to respect statistical independence without exception. The consequences of this statement are being explored currently and will be reported elsewhere.

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