Acta Mechanica (1994) [Suppl] 4: 113–123

Multiplicative models for turbulent energy dissipation*

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Summary. We consider models for describing the intermittent distribution of the energy dissipation rate per unit mass, ε, in high-Reynolds-number turbulent flows. These models are based on a physical picture in which (in one-dimensional space) an eddy of scale r breaks into b smaller eddies of scale r/b. The energy flux across scales of size r is $r\varepsilon_r$, where ε_r is the average of ε over a linear interval of size r. This energy flux can be written as the product of factors called multipliers. We discuss some properties of the distribution of multipliers. Using measured multiplier distributions obtained from atmospheric surface layer data on ε, we show that quasi-deterministic models (multiplicative models) can be developed on a rational basis for multipliers with bases $b = 2$ and 3 (that is, binary and tertiary breakdown processes). This formalism allows a unified understanding of some apparently unrelated previous work, and its simplicity permits the derivation of explicit analytic expressions for quantities such as the probability density function of $r\varepsilon_r$, which agree very well with measurements. Other related applications of multiplier distributions are presented. The limitations of this approach are discussed when bases larger than three are invoked.

1 Introduction

A Gaussian process is completely described in a statistical sense by its mean and standard deviation. It is conceivable that a nearly Gaussian process can be described well by its first few moments – at least well enough for most purposes. This is the situation with respect to velocity or temperature traces obtained in high-Reynolds-number fully turbulent flows not too close to physical boundaries. On the other hand, the situation is quite different for quantities such as the velocity derivatives and the energy dissipation rate. In high-Reynolds-number turbulence where the small-scale motion is isotropic to a good approximation [1], the average energy dissipation rate per unit mass can be written as $\langle\varepsilon\rangle = 15\nu\langle(\partial u/\partial x)^2\rangle$, where the angular brackets indicate averages, and u is the x-component of the velocity and ν is the fluid viscosity. ∂t is often assumed, as we shall do, that $(\partial u/\partial x)^2$ is a representative component of ε instantaneously. Figure 1 is a one-dimensional section through the field of ε in the atmospheric surface layer a few meters over land. In contrast to Gaussian or nearly Gaussian processes, information about the first few low-order moments does not describe the signal in any detail. Peaks which are hundreds of times the mean are not uncommon, and the signal is at other times of very low amplitude; this strongly intermittent character is a typical property of ε in high-Reynolds-number turbulence. It is becoming increasingly clear that intermittency plays a fundamental role in the understanding and modeling of turbulent flows. The intermittency has important implications also in contexts such as the structure of turbulent flames.

In the last few years, much work based on multifractals has occurred on the description and modeling of the intermittent character (and other similar characteristics) of the energy

* This is the enlarged version of the text to appear in the Proceedings of the International Congress of Theoretical and Applied Mechanics, Haifa, 1992.

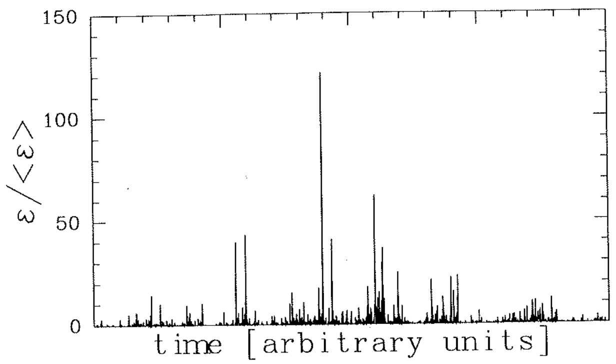


Fig. 1. A typical signal of a representative component of ε, namely $\varepsilon' \sim (du/dx)^2 \sim (du/dt)^2$, normalized by its mean. Here, u is the velocity fluctuation in the direction x of the mean velocity U. In writing the last step of the above approximation, it has been assumed that Taylor's frozen flow hypothesis, namely that the spatial derivative can be approximated by the temporal derivative, holds. The velocity fluctuation u was obtained by a hot-wire mounted on a pole 6 m above the ground level over a wheat canopy. The microscale Reynolds number is of the order 2500. The resolution is of the order of the Kolmogorov scale

dissipation rate. For a summary, see [2]. In [3–5], several simple quasi-deterministic multifractal models were shown to describe the statistical properties of the energy dissipation rate quite accurately. Here, we provide a basis for developing such simplified models and show that it brings unity to some of the existing and seemingly unrelated work.

The energy dissipation rate is a positive definite quantity which is additive (in the sense that ε over two non-overlapping intervals equals the sum of ε values distributed over the sum of the two intervals). In this sense, it is convenient to think of ε as a measure distributed on an interval.

2 Multiplicative processes and multiplier distributions

The thought behind the models to be described is that the nonlinear processes occurring in the so-called inertial range (i.e., the scale range much larger than dissipative scales and much smaller than an external length scale such as the integral scale) may be abstracted by a breakdown process in which each eddy of size r subdivides into b pieces of size r/b. In this process, the energy flux per unit mass is redistributed in some unequal fashion without loss among b sub-eddies. This unequal distribution among sub-eddies is the heart of the observed intermittency. While, in reality, turbulence dynamics involves vortex stretching at many scales, the eddy breakdown processes do contain some essential physics of the inertial range.

In this model, the k-th sub-eddy ($1 \leq k \leq b$) will receive a fraction $M_k^{(r)}$ of the energy flux carried by its parent eddy; the conservation of the energy flux implies that $M_1^{(r)} + \cdots + M_b^{(r)} = 1$. In this manner, if we traced the history of all forefathers (up to the external scale L) of a given eddy at scale r, we would find that the energy flux $r\varepsilon_r$ being transferred to this eddy is related to the energy flux at the scale L as

$$r\varepsilon_r = L\varepsilon_L M^{(L)} M^{(L/b)} \ldots M^{(L/b^{n-1})} \tag{1}$$

where $r/L = b^{-n}$, and $\varepsilon_L = \langle\varepsilon\rangle$ (since the average over the scale L is not distinguishable from the global average indicated by $\langle\,\rangle$).

At different scales, the multipliers are assumed to be independent stochastic variables. Since the energy flux, as it cascades down to smaller scales, is ultimately converted into energy dissipation, the two quantities have equal averages and, for present purposes, will be considered identical. The dynamics described above is, of course, much simpler than the reality. For instance, it is not obvious what the appropriate value of b (that is, the base for the cascade process) must be, or whether it remains the same from one step of the cascade to another. In spite of this ambiguity, it is clear that if the breakdown process has statistical scale-similarity, the histograms of multipliers $M^{(r)}$ should be identical at each step of the subdivision or cascade, i.e., independent of r.

How can we obtain the distribution of multipliers? Consider a long data string of ε distributed over an interval which is N integral scales in extent, N being some large integer. Divide each interval of size L into b equal-sized sub-intervals, and obtain the ratios of the measures (i.e. the integrals of ε) in each of the sub-intervals to that in the entire interval. These ratios, which are what we called multipliers, are clearly positive (since ε is positive definite) and lie between zero and unity. Subdivide each sub-interval into b pieces as before, and repeat the procedure. Proceeding with further subdivisions, there will be $N \cdot b^n$ multipliers $M_i^{(r)}$ at the n-th subsequent level, where each sub-interval is of size $r/L = b^{-n}$. Construct the histogram of the multipliers at each level, and repeat the procedure until the smallest sub-interval reached is of the order of the Kolmogorov scale $\eta = L(\nu^3/\langle\varepsilon\rangle)^{1/4}$.

The probability density $P(M)$ of the multipliers M – here and subsequently, we omit the indices on the $M_i^{(r)}$ and denote them simply by M – has been obtained from measurement for different stages of subdivision of the interval. Since the value of the base b is not known *a priori*, Chhabra and Sreenivasan [5] obtained $P(M)$ for various bases. Figure 2 shows the results for $b = 2, 3$ and 5. The shape of each of the distributions is invariant over a certain range of scales, suggesting that some type of self-similarity occurs in this scale range, whatever the assumed base. This range over which $P(M)$ is self-similar agrees quite well with the inertial range determined by

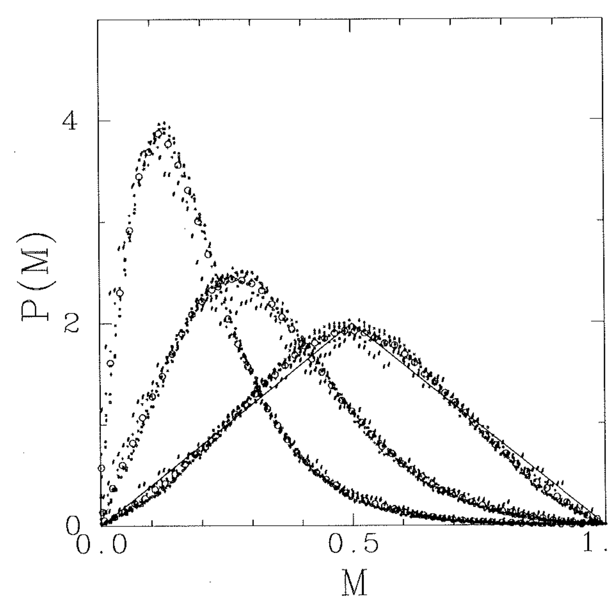


Fig. 2. Multiplier distributions $P(M)$ for bases (from right to left) $b = 2, 3$ and 5. The larger symbols show averaged multiplier distribution, which are the mean of multiplier distributions obtained by comparing measure in boxes of size m to those of $m*b$, where m ranged from 50 to 1000 in units corresponding roughly to the Kolmogorov scale. The smaller symbols show the distributions obtained for $m = 50, 80, 150, 200, 400$ and 1000. The solid line is the triangular approximation to the binary case. The figure is adapted from [5]

the scaling of the energy spectrum and structure functions, and covers about a decade and a half. The larger symbols show an average over steps involving comparisons between boxes of size m and those of size $m*b$, where m ranged from 50 to 1 000 in units of sampling intervals. (For the very smallest scales, the distributions have a concave shape. This concavity is related to the divergence of moments [7] and will be discussed elsewhere. For very large scales, multiplier distributions approach a delta function centered around 0.5, as would be the case for random measures.)

A disadvantage of $P(M)$ is that it is base-dependent. However, if the cascades giving rise to the observed intermittency are randomly multiplicative, then the multiplier distributions corresponding to different bases are related by convolution, and one can scale out this base-dependency [5]. In particular, for any two bases 'a' and 'b', we have

$$\log \langle (M_a)^q \rangle / \log (a) = \log \langle (M_b)^q \rangle / \log (b) . \tag{2}$$

The scale-invariant multiplier distributions obtained in Fig. 2 are fundamental to the understanding of the observed scaling in turbulence. One can compute from them not only the asymptotic scaling properties of ε_r, but also finite-size fluctuations of scaling properties [5]. In order to understand this, it is helpful to note the relation between the multiplier distributions of Fig. 2 and the multifractal description of turbulence. In the latter, $r\varepsilon_r$ is assumed to scale as

$$r\varepsilon_r = L\varepsilon_L (r/L)^{\alpha} , \tag{3}$$

where the exponent α (the so-called Hölder exponent) depends on the spatial coordinate position.

The probability of finding a scaling index α, $P(\alpha)$, also scales as

$$P(\alpha) \sim (r/L)^{d - f_d(\alpha)} , \tag{4}$$

where d is the dimension of the space under consideration, and $f_d(\alpha)$ is called the multifractal spectrum [8], and is dependent on d. For one dimensional cuts (as those of Fig. 1), $d = 1$ and we drop the suffix on $f(\alpha)$. Then the moments of $r\varepsilon_r$ can be computed from the above relations as

$$\langle (r\varepsilon_r)^q \rangle = (L\varepsilon_L)^q \int d\alpha (r/L)^{q\alpha + (1 - f(\alpha))} \sim (L\varepsilon_L)^q \, (r/L)^{\tau(q) + 1} \tag{5}$$

where $\tau(q) = \min_\alpha \{q\alpha - f(\alpha)\}$, and it was assumed that $r \ll L$.

The $f(\alpha)$ function can also be easily derived from the multiplier distribution. In [5], it was shown that the $f(\alpha)$ curves computed from the distributions for different bases b were in good agreement with each other as well as with those obtained from direct methods such as box-counting [4]. This agreement indicates the existence of a probabilistic cascade where no single base is preferred.

Incidentally, a good approximation for $P(M)$ in the binary case (i.e., the base $b = 2$) is the triangular distribution shown in Fig. 2. In [5] it was shown that the $f(\alpha)$ function for this model is in agreement with that obtained directly [4]. Further, the triangular model displays the correct behavior with respect to sample-to-sample fluctuations in $f(\alpha)$, and reproduces the stretched exponential tails, $P(\varepsilon) \sim \exp(-\beta(\varepsilon)^{1/2})$, observed in [4, 9] for the probability distribution of ε.

3 Simple quasi-deterministic models

The multiplier distributions shown in Fig. 2 are extracted directly from the experiment and their analytical forms are yet to be found from the theory. The question meanwhile is a simple representation of these distributions in a way that permits one to evaluate most of the measured

properties accurately. The goal is to seek models that are simple enough to be tractable mathematically and realistic enough to represent the spirit of the underlying physics. A simple possibility is the p-model [3], which is a model for a binary cascade ($b = 2$). We first discuss the p-model and show how it can be obtained as a rational approximation to the measured multiplier distribution for the binary case. We will then discuss how models in the same spirit can be obtained for the tertiary case ($b = 3$). The limitations of the procedure for high order subdivisions ($b > 3$) will be highlighted.

From a physical point of view, the cascade process with $b = 2$ can be thought of as the break-up of a structure (the parent eddy, scale or structure) into two sub-structures. For the one-dimensional case corresponding to Fig. 1, a pertinent question is the following: is there any difference between the left and right offsprings in terms of the energy flux they receive from the parent structure? One can determine experimentally that left and right are statistically indistinguishable. (This is not true for the velocity signal itself, as can be concluded from Kolmogorov's 4/5 law [10].) Now, for the sake of simplicity and modeling, let us assume that one of the two sub-eddies always receives a fixed fraction p of the energy contained in the parent eddy; naturally, the other will receive $1 - p$. In this sense, this model is deterministic. However, it is only quasi-deterministic in the sense that either one of the two eddies could receive the fraction p; because of the left-right symmetry mentioned above, a given piece will receive p as often as $1 - p$. Then, the multiplier distribution for the p-model can be written in terms of delta functions as

$$P_{b=2}(M) = \frac{\delta(M - p) + \delta\big(M - (1 - p)\big)}{2} . \tag{6}$$

If $p = 1/2$, there is no intermittency and the situation corresponds to Kolmogorov's 1941 theory [1]. To obtain intermittency, we should have a value of p different from 1/2.

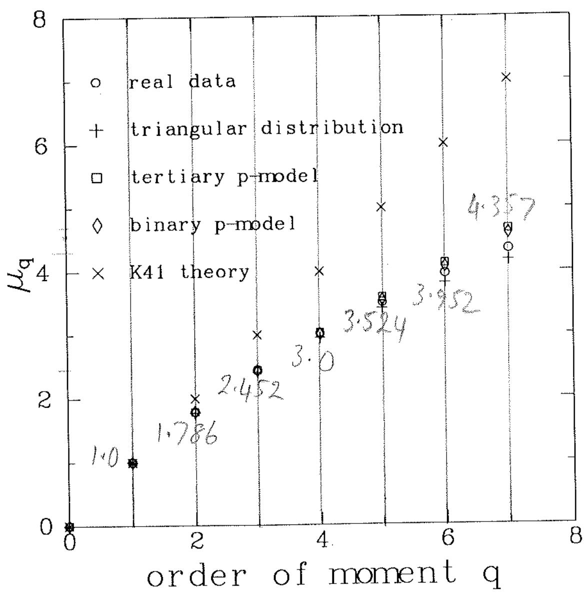
How can we determine the value of p? A natural way is to match the moments of $P_{b=2}(M)$ from Eq. (6) with those of the real $P(M)$. For both distributions, the zero-order moment (normalization) and the first-order moment (mean value) coincide trivially, and are 1 and 1/2, respectively. The first non-trivial condition is obtained by matching the second order moment. By this matching condition, we obtain the value $p = 0.697$, or $1 - p = 0.303$.

In a completely different way, it was shown in [3] that a good fit to the $f(\alpha)$ curve can be achieved if a binary cascade with parameters $p = 0.7$ (and $1 - p = 0.3$) is assumed as the energy transfer mechanism. In [3], the numbers 0.7 and 0.3 appeared a little mysteriously as the effective fractions of energy split in the breakdown process. Even though it was recognized that a multiplicity of multipliers may occur (the multiplier distribution had not been measured at that time), it was surprising that such a simple model could do so well. We now have an explanation, which is that the binary p-model is an approximation that fits the zero-th, first and the second moments of the measured multiplier distributions. It fortunately turns out that high-order moments computed for the p-model also agree with those computed for the real data (see Fig. 3). The binary p-model can now be considered as derived from the real multiplier distribution.

We now discuss a general scheme for developing for all $b \neq 2$ quasi-deterministic models of the type developed for the binary case. Again, we attempt to do this by matching moments. The general multiplier distribution for any b in the p-model scheme is

$$P_b(M) = \frac{1}{b} \sum_{i=1}^{b} \delta(M - p_i) \tag{7}$$

μ_q (novi) $= q - \mu_q$ (the present)



q	in this notation ↓ μ_q	Novikov's notation ↓ μ_q	$\mu_q/q-1$
2	1.786	0.214	0.214
3	2.452	0.548	0.274
4	3	1.0	0.333
5	3.524	1.476	0.369
6	3.952	2.048	0.410
7	4.357	2.643	0.441

Fig. 3. A comparison between the scaling exponents μ_q (Eq. (16)) computed from the measured multiplier distributions and those computed for the different models considered in the text. Experimental data were obtained from a record length of 810000 data points. The convergence of moments was reasonable; for example, in the last half decade of the record length, the variations observed were smaller (in the log scale) than the symbol size

where

$$\sum_{i=1}^{b} p_i = 1, \quad (0 \leq pi \leq 1). \tag{8}$$

We may now equate the moments of $P_b(M)$ to the moments of the real multiplier distributions. Since multiplier distributions for any base yield the same results, we may take the distribution corresponding to the binary cascade of Fig. 2. Computing the moments $\langle M^q \rangle$ from it, we are led to the equations

$$\sum_{i=1}^{b} p_i^q = b\langle M^q \rangle^{\log(b)/\log(2)}, \quad q = 1, 2, \ldots, b. \tag{9}$$

This is a system of b equations with b number of p_i's to be determined. Using Girard's rule [11], it is easy to find a polynomial of degree b whose roots are the desired p_i's. The problem thus reduces to the determination of the roots of the polynomial. It turns out *a posteriori* that this problem has physical solutions only for $b = 2$ and $b = 3$: for larger values of b, some of the roots turn out to be complex, and have no physical meaning. The values of p_i for the tertiary cascade ($b = 3$) are $p_1 = 0.155$, $p_2 = 0.283$ and $p_3 = 0.562$. We designate this as the tertiary p-model. The classical Kolmogorov theory for a tertiary breakup scheme yields $p_1 = p_2 = p_3 = 1/3$. Figure 3 shows that the comparison with the experimental values is good.

It is interesting to note that Viczek and Barabasi [12] empirically devised the tertiary p-model without any reference to the multiplier distribution. Briefly, they discussed an iterative scheme for generating a velocity signal that possessed many features of the actual turbulent velocity time trace. The starting step picks at random one of the two generators $z_c(x)$ (Fig. 4a) and $z_d(x)$ (Fig. 4b) shown in Fig. 4. The magnitudes c and d will be specified shortly. Each of the 3 linear

segments resulting after the first step is randomly replaced with a rescaled version of the generators, so that at each step the function is continuous. If the slope of the segment to be replaced is positive, we use with equal probability $z_c(x)$ or $z_d(x)$. If it is negative, we use with equal probability $1 - z_c(x)$ or $1 - z_d(x)$. This is illustrated in Fig. 4c.

This scheme represents, by construction, a tertiary process. Denote by $\Delta u(r)$ the velocity increment $u(x + r) - u(x)$ across a distance $r = L3^{-k}$. At the step $k + 1$, the velocity difference across an interval $r/3$ (included in the interval $[x, x + r]$), can be one of the following three possibilities:

$$|\Delta u(r/3)| = |\Delta u(r)| \begin{cases} c \\ c - d \\ 1 - d \end{cases} \tag{10}$$

where the parameters c and d are defined in Fig. 4. Knowing that $\langle |\Delta u(r)|^q \rangle \sim (r/L)^{\zeta_q}$, Viczek and Barabasi computed ζ_q for their model, and found that in order to match experiments, the values of c and d had to be $c = 0.67$ and $d = 0.13$.

Interestingly, our tertiary p-model yields a very similar result. In effect, according with Kolmogorov's refined similarity hypotheses [13] — see also Section 4 — $\Delta u(r) \sim (r\varepsilon_r)^{1/3}$. For our

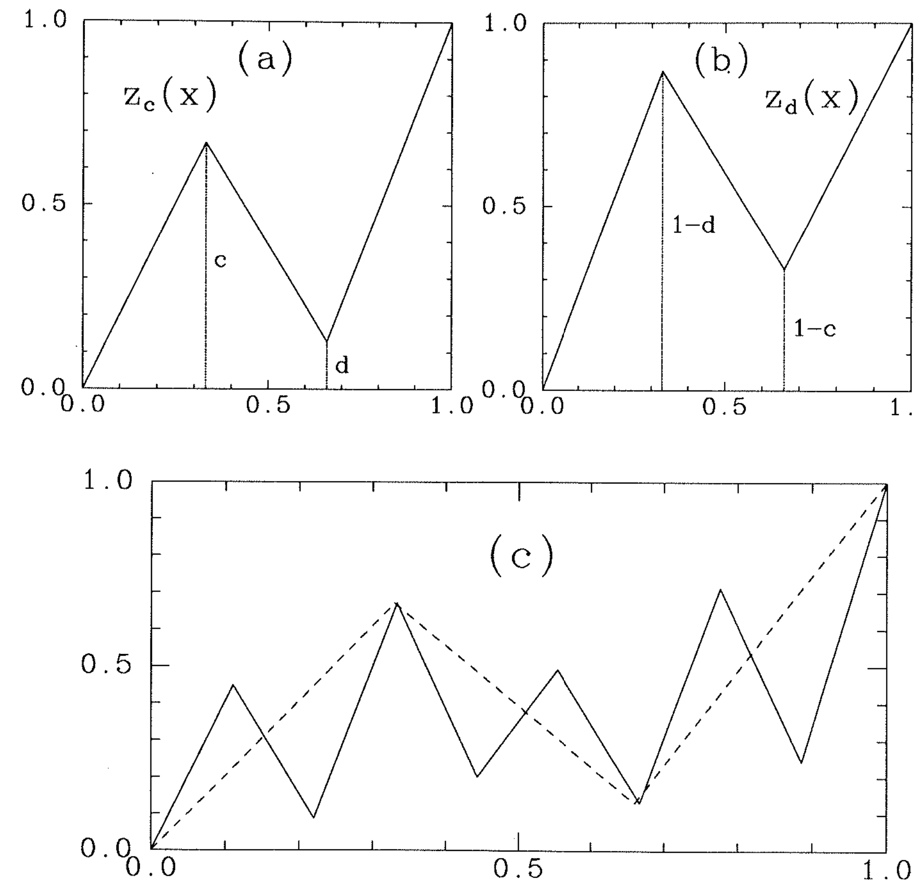


Fig. 4. Generators $z_c(x)$ and $z_d(x)$ used for the iterative scheme discussed in [12] to model a velocity trace (see **a** and **b**). At each step, a linear segment is replaced, with equal probability, by a rescaled version of $z_c(x)$ or $z_d(x)$ if the segment derivative is positive, and by $1 - z_c(x)$ or $1 - z_d(x)$ if the segment derivative is negative. **c** Example of the scheme at the first step (dashed line) and the second step (solid line)

multiplicative model, $r\varepsilon_r = L\varepsilon_L M_1 \ldots M_n$, and we conclude from Eq. (10) that c, $c-d$ and $1-d$ have to be related with the cubic root of our multipliers $p_1{}^{1/3}$, $p_2{}^{1/3}$ and $p_3{}^{1/3}$ for the tertiary p-model. These values are 0.66, 0.54 and 0.82, while Viczek and Barabasi's values for c, $c-d$ and $1-d$ are 0.67, 0.5 and 0.83. It is important to realize that the two methods of finding c and d, and the p's, are completely different. Hence, as in the case of the binary p-model, we note that the parameters c and d can be derived from the multiplier distribution, although such was not its original derivation.

Returning to the generation of quasi-deterministic models, it is clear that increasing the number of p-parameters would enable us to fit the measurements better and better. While a model with too many parameters is not very useful, it would be desirable to be able to fit moments up to order 8 – the reason being that the three-dimensional analogue of the one-dimensional binary breakdown $b = 2$ is the case of a cube breaking into 8 pieces ($b = 8$). We mentioned previously that quasi-deterministic p-models can be constructed only up to $b = 3$. While this tells us about the impossibility for quasi-deterministic models to fit experiments with any *predetermined accuracy*, it does not mean that working quasi-deterministic models cannot be constructed for $b > 3$. For example, for the three dimensional binary cascade, we can write a typical multiplier as

$$M^{(3D)} = p_1{}^{(1D)} p_2{}^{(1D)} p_3{}^{(1D)} \tag{11}$$

where $p_j{}^{(1D)}$ are the result of a binary cascade for one dimension, and is equal to 0.3 or 0.7. Then

$$P(M^{(3D)}) = \frac{1}{8}\{\delta(M^{(3D)} - 0.7^3) + \delta(M^{(3D)} - 0.3^3)\} + \frac{3}{8}\{\delta(M^{(3D)} - 0.3^2 0.7) + \delta(M^{(3D)} - 0.7^2 0.3)\}. \tag{12}$$

Although this distribution can exactly fit only up to the second moment of the real multiplier distribution, the agreement with higher order moments is again good.

Other models have also been proposed. For example, Novikov [14] proposed a uniform distribution for $P(M)$. At the time the model was proposed, the multiplier distribution had not been obtained experimentally. It is now clear, however, that a uniform distribution is not a good model for any of the curves in Fig. 2. For example, a good approximation to the binary cascade is the triangular distribution shown by a solid line in Fig. 2.

4 Further applications

In the previous Section we discussed the application of p-models towards understanding some multifractal aspects of turbulence as well as a velocity model. A further application to be discussed here is the calculation of the probability density of $r\varepsilon_r$. By the definition of multipliers,

$$r\varepsilon_r = L\varepsilon_L \prod_{i=1}^{n} M_i, \tag{13}$$

where $n(r) = \log_b(r/L)$, and L (as before) is the large-eddy size or the integral scale. Within the binary p-model scheme ($b = 2$), $x = r\varepsilon_r / L\varepsilon_L$ can take the discrete values $p^k(1-p)^{n-k}$, where $p = 0.3$ and $0 \leqq k \leqq n$. The probability of occurrence of a given k is simply $2^{-n} n!/(k!(n-k)!)$. Replacing factorials by gamma functions (to allow for continuous k), the probability density

function P for the ratio x is

$$P(x) = \frac{A}{x}\frac{r}{L}\frac{\Gamma(\log_2(r/L)+1)}{\Gamma(k(x)+1)\,\Gamma(\log_2(r/L)-k(x)+1)}, \tag{14}$$

where A is a normalization constant of order unity (and would be equal to unity in the discrete case), Γ is the gamma function and

$$k(x) = \frac{\log\left(x(r/L)^{\log_2(1-p)}\right)}{\log\left(p/(1-p)\right)}. \tag{15}$$

Figure 5 shows a comparison between experimentally measured distribution of $r\varepsilon_r$ (normalized with its standard deviation) and the prediction of Eq. (14). The experimental data used are the same as those used to compute the multiplier distributions of Fig. 2. We found that to fit the data at $r/\eta = 75$ we had to use $n(r = 75\eta) = 14$ (Fig. 4a). For $r/\eta = 1\,200$ (Fig. 4b), we computed $n(r = 1\,200\eta) = n(r = 75) - \log_2(1\,200/75) = 10$. The agreement with experiment is good.

Another application is in computing the exponents μ_q, defined as

$$\langle (r\varepsilon_r)^q \rangle \sim (r/L)^{\mu_q}. \tag{16}$$

The general result is that $\mu_q = -\log_b \langle M^q \rangle$. The numerical value of μ_q depends upon the distribution used to compute $\langle M^q \rangle$. The results for the different models considered in this

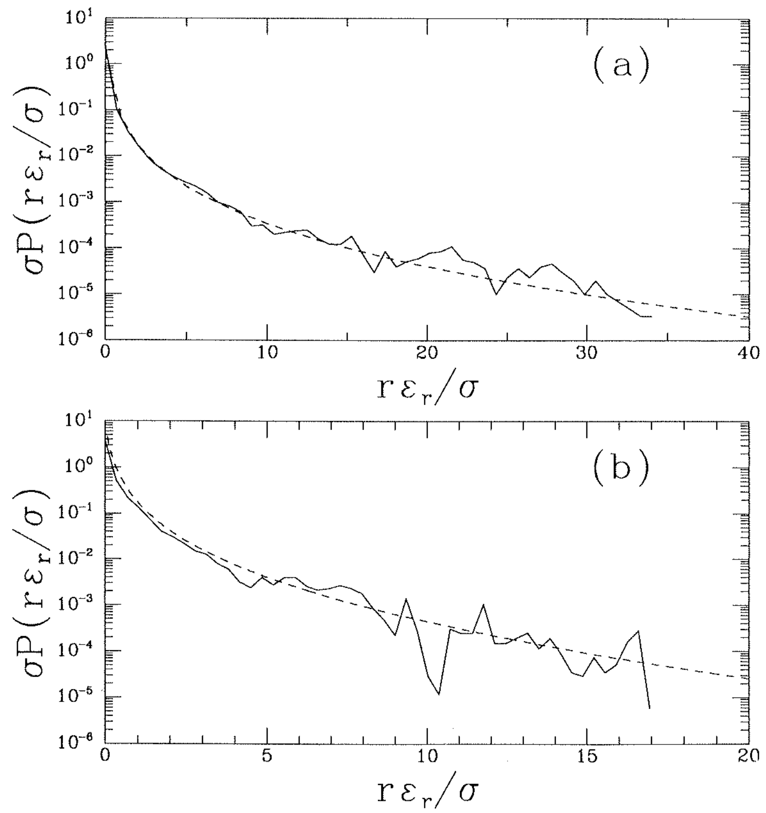


Fig. 5. Experimental (solid line) and binary p-model prediction (Eq. (14), dashed line) for the probability density function of $r\varepsilon_r$ for two values of the separation distance r, normalized by their respective standard deviations. **a** Separation distance $r = 75\eta$, number of steps in the cascade $= 14$. **b** Separation distance $r = 1\,200\eta$, number of steps in the cascade $= 10$

paper are

$$\mu_q = \begin{cases} -\log_2 \left\{ 4 \left(\frac{1}{2^{q+1}} \right) \frac{1}{(q+1)(q+2)} \right\} & \text{triangular distribution} \\ -\log_2 [(0.3^q + 0.7^q)/2] & \text{binary } p\text{-model} \\ -\log_3 [(0.155^q + 0.283^q + 0.562^q)/3] & \text{tertiary } p\text{-model}. \end{cases} \tag{17}$$

These different results have already been plotted along with the experimental values in Fig. 3. It is worth remembering that, although the binary and tertiary p-models, respectively, are generated to possess the first three and four moments correctly, they agree well with the measured moments up to, say, order 7.

The probability density functions of the velocity increments can also be computed. On using the second refined similarity hypothesis [13], the velocity increments can be written as

$$\Delta u(r) = V(r\varepsilon_r)^{1/3}, \tag{18}$$

where V is a universal stochastic variable independent of r and $r\varepsilon_r$. In [15], we obtained the probability density of V. The probability density of Δu can be computed if one of the previously described models is assumed for $r\varepsilon_r$. In [9], those probability density functions were computed using the binary p-model, and were found to be in good agreement with the data.

5 Conclusions

The multiplier distribution is a basic tool for understanding many of the scale-similar properties of energy dissipation in turbulence. In the absence of an *ab initio* theory that yields these distributions in a deductive way, analytical progress can be made only by modeling them with reasonable schemes. Here, we have summarized the attempts made in the last few years and mentioned a few applications of this work. We have also shown that the quasi-deterministic models described here, deductively derived from the real multiplier distribution, can be used to bring unity to various scattered models of small scale turbulence and the phenomenon of intermittency. Even though the physics suggested by these quasi-deterministic models is a caricature of more complicated nonlinear processes, they are devised to provide quantitative information. As we have shown in this paper, they accomplish their aim far better than their simplicity might suggest.

It is a pleasure to dedicate this paper to Professor J. Zierep on the occasion of his sixty-fifth birthday.

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