Probability Density of Velocity Increments in Turbulent Flows

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Measurements have been made of the probability density function (PDF) of velocity increments $\Delta u(r)$ for a wide range of separation distances r. Stretched exponentials provide good working approximations to the tails of the PDF. The stretching exponent varies monotonically from 0.5 for r in the dissipation range to 2 for r in the integral scale range. Theoretical forms based on multifractal notions of turbulence agree well with the measured PDFs. When the largest scales in the velocity u are filtered out, the PDF of $\Delta u(r)$ becomes symmetric and, for large r, close to exponential.

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Much work in the turbulence literature [1-4] has been devoted to the determination of the scaling properties of the structure functions $\langle[\Delta u(r)]^n\rangle$, where n is a positive integer and $\Delta u(r)$ is the velocity increment between two spatial locations which are a distance r apart. It has recently [5] been emphasized that a better strategy may be to focus on the probability density functions (PDFs) of $\Delta u(r)$, $p_{\Delta u}(\Delta u)$, rather than on the collection of moments. Accordingly, this Letter is concerned with the PDFs of $\Delta u(r)$ and of velocity derivatives, and has three purposes. First, it provides experimental data on $p_{\Delta u}(\Delta u(r))$ for a large range of separation distances r spanning the dissipation range on one end and the integral scale range on the other. The data show that the PDFs are square-root exponential for r in the dissipation range and Gaussian for r on the order of the correlation (or the integral) scale. Second, it provides for the PDFs a theoretical expression based on the multifractal picture for turbulence dynamics. The third aspect is related to the exact result known for velocity increments [6], namely, that in locally homogeneous turbulence, the third-order structure function in the inertial range obeys the relation

$$\langle[\Delta u(r)]^3\rangle = -\tfrac{4}{5}\langle\varepsilon\rangle r\,. \tag{1}$$

While the implications of the nonzero value of $\langle[\Delta u(r)]^3\rangle$ have not been understood fully, it has been shown recently [7] that the removal of the largest scales by high-pass filtering renders the PDF of Δu symmetric. This Letter provides experimental results on the $p_{\Delta u}$ when the low-frequency (or large scale) components of u have been filtered out. For complementary results on PDFs of velocity derivatives and increments, see Refs. [8] and [9]. We shall subsequently return to some of this work.

Measurements were made in the atmospheric surface layer about 6 m above a wheat canopy in the Connecticut Agricultural Research Station. Data were also acquired about 2 m above the roof of a four-story building. The laboratory data were acquired at a height of 0.2δ, where δ is the thickness of the boundary layer, over a smooth flat plate. The boundary-layer-thickness Reynolds number at the measuring station was 32000. Velocity fluctuations were measured using the standard hot-wire (5 μm diam, 0.6 mm length) velocimeter operated in the constant-temperature mode on a DISA 55M01 anemometer. The anemometer voltage was digitized on a 12-bit digitizer and linearized before further processing. Velocity derivatives were obtained by central differencing of the data. Taylor's frozen flow hypothesis was used in interpreting time intervals as space intervals. The precise limitations of this hypothesis are unclear (in spite of much work), especially for the tails of the PDF, but it should be noted that the mean convection velocity in the present experiments was about 15 times larger than the standard deviation of the fluctuating velocity. For obtaining the PDF of the filtered data, a linear phase filter with excellent cutoff characteristics [10] was used.

We have fitted stretched exponentials, $p_{\Delta u} \sim \exp(-\alpha|\Delta u|^m)$, to the tails of $p_{\Delta u}(\Delta u)$. The inset to Fig. 1 demonstrates, for one arbitrarily chosen r, that

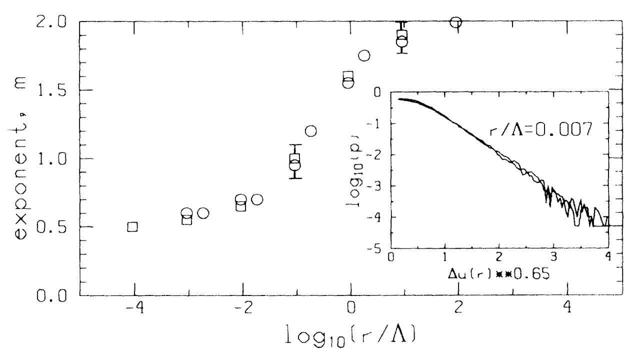

FIG. 1. The experimentally determined stretching exponent m in $p_{\Delta u}(\Delta u(r)) \sim \exp[-\alpha(\Delta u)^m]$, plotted as a function of r/Λ. ○ and □ are for two different sets of atmospheric data. The Taylor microscale Reynolds number R_λ was on the order of 1500 for both. The laboratory data, not plotted here, show a similar trend and roughly coincide with the atmospheric data. $\Delta u(r)$ is obtained by taking the velocity differences separated by a time difference Δt, and interpreting $r = -\Delta t\,U$, where U is the mean velocity of the flow at the measurement station (Taylor's frozen flow hypothesis). The integral scale Λ was determined by obtaining the area under the autocorrelation function of u, and converting it to a length scale by Taylor's hypothesis. Inset: The stretched exponential is a good approximation to typical experimental data for both sides of the distribution. In this figure and others in this paper, Δu is normalized by its root-mean-square value.

there is an extensive region of the PDF to which a stretched-exponential fit is good. It follows from Eq. (1) that there must be a certain asymmetry between the two tails of the distribution, but this asymmetry is not very large. In fact, Fig. 1 shows that the differences between the two tails of the distribution, insofar as they relate to the stretched-exponential fits, are small. We shall therefore momentarily ignore this asymmetry and return to it later. The empirically determined stretching exponents are plotted in Fig. 1 as a function of r for two sets of atmospheric data. While square-root-exponential fits are good for r in the dissipation range, Gaussian fits are appropriate for $r \simeq L_0 - 10\Lambda$, where Λ is the autocorrelation length scale of u (see caption to Fig. 1). This latter result is not surprising because velocities at two widely separated points become independent of each other. The exponent m increases monotonically through the inertial range. Laboratory data, not displayed here, show a similar behavior.

It should be noted that a somewhat similar effort for temperature increments in a Rayleigh-Bénard convection experiment has been made in Ref. [11].

In a recent paper [9], Benzi *et al.* derived an expression for the PDF of velocity increments. They assumed that $\Delta u(r)$, for r in the inertial range, is given by the random β model [12],

$$\Delta u(r) = \Delta u_0 (r/L)^{1/3} \prod_{i=1}^{n} \beta_i^{-1/3}, \tag{2}$$

where Δu_0 is the characteristic velocity increment on the macroscopic length L_0, and the β_i's are identically distributed independent random variables. Benzi *et al.* [9] used a special case in which the probability density of β was assumed to be given by

$$p_\beta(\beta) = a\delta(1-\beta) + (1-a)\delta(B-\beta), \tag{3}$$

with $a = \frac{7}{8}$ and $B = \frac{1}{2}$, in conformity with experiments [2,3]. Under the further assumption that Δu_0 is normally distributed (see Fig. 1), Benzi *et al.* [9] showed that the PDF of Δu is given by

$$p_{\Delta u}(\Delta u(r)) = \sum_{k=0}^{n} C_k^n a^{n-k}(1-a)^k B^k \frac{1}{(2\pi\sigma_{k;n}^2)^{1/2}} \exp\left(-\frac{\Delta u^2}{2\sigma_{k;n}^2}\right) + \{1 - [a + (1-a)B]^n\}\delta(\Delta u), \tag{4a}$$

$$\sigma_{k;n} = \frac{\langle \Delta u_0^2 \rangle^{1/2} (r/L)^{1/3}}{B^{k/3}}. \tag{4b}$$

Here, n is the number of steps assumed to occur in a cascade before reaching a scale r and is given, for a binary cascade, by $n = \log_2(L_0/r)$. Notice that the second term on the right-hand side of Eq. (4a) accounts for inactive eddies, i.e., spatial regions where the dissipation of energy is zero. Away from $\Delta u = 0$, the first term on the right-hand side of Eq. (4a) is the only contribution to the total PDF.

An alternative expression can be obtained for $p_{\Delta u}(\Delta u(r))$ as follows. Following Kolmogorov [13] we assume that for r in the inertial range of scales

$$\Delta u(r) = v(r\varepsilon_r)^{1/3}, \tag{5}$$

where v is a "universal" stochastic variable, and $r\varepsilon_r$ is the total energy dissipation in the linear piece of size r. It is clear that within this framework [14] any reasonable model for ε_r and v will also yield a reasonable model for the PDF of $\Delta u(r)$. A convenient model [15] for the energy dissipation is one in which the average energy flux summed over any box of size $r/L_0 = 2^{-n}$ can be written as

$$r\varepsilon_r = L_0 \varepsilon_0 \prod_{i=1}^{n} m_i, \tag{6}$$

where the multipliers m_i are identically distributed independent random variables. Here $\varepsilon_0 L_0$ is the typical energy dissipation contained in an eddy at the macroscopic scale (or, equivalently, the total energy flux across scales). Noting that $\langle \Delta u_0^2 \rangle \sim (L_0\varepsilon_0)^{2/3}$, we write

$$\Delta u(r) = v \langle \Delta u_0^2 \rangle^{1/2} \prod_{i=1}^{n} m_i^{1/3}. \tag{7}$$

Recalling again from Fig. 1 that $p_{\Delta u}(\Delta u_0)$ is Gaussian (with mean 0 and variance $\langle \Delta u_0^2 \rangle^{1/2}$), we obtain from Eq. (7), with $n=0$, that *the universal stochastic variable v has a normal distribution with zero mean and unity variance.* In stating this result, we are stretching the validity of Eq. (5) all the way to the integral scales of motion.

Now, Meneveau and Sreenivasan [16] have shown that the following simple model (the so-called p model) for the probability density of m is adequate for most purposes:

$$p_m(m) = 0.5[\delta(m-M) + \delta(m-(1-M))] \tag{8}$$

with $M = 0.3$. Using the normality of v and Eq. (8), we get from Eq. (7) that

$$p_{\Delta u}(\Delta u(r)) = \sum_{k=0}^{n} C_k^n 2^{-n} \frac{1}{(2\pi\sigma_{k;n}^2)^{1/2}} \exp\left(-\frac{\Delta u^2}{2\sigma_{k;n}^2}\right), \tag{9a}$$

$$\sigma_{k;n} = \langle \Delta u_0^2 \rangle^{1/2} M^{k/3} (1-M)^{(n-k)/3}. \tag{9b}$$

Figure 2 shows comparisons between the experimental data for two values of r ($r/\Lambda = 0.009$ in the inertial range and $r/\Lambda = 0.9$) and the theoretical results of Eqs. (4) and (9). The appropriate values of n to be used in Eqs. (4) and (9) are determined by the relation $n = \log_2(L_0/r)$.

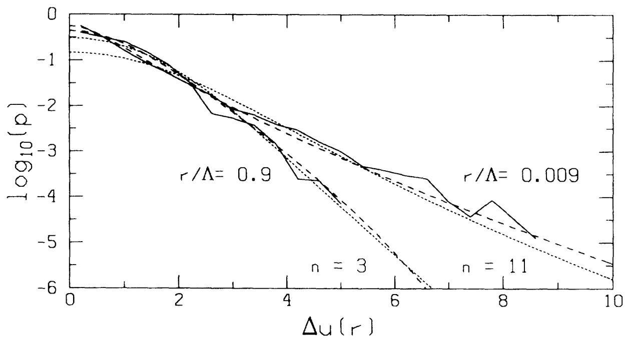

FIG. 2. A comparison between the experimental data for $p_{\Delta u}(\Delta u)$ for two values of r, and Eq. (9) based on the p model (dashed line) and Eq. (4) based on the random β model (dotted line). The formula based on the random β model tends to underestimate the PDF of Δu close to 0.

Both expressions fit the data reasonably well, but the theoretical curve for the random β model falls below the experimental data in the region near $\Delta u = 0$. This is so because in this model the active zones ($\Delta u \neq 0$) tend to become more sparse as n increases.

Equation (9) expresses the probability of finding a velocity increment Δu at a fixed scale r. The probability of finding a "gradient" $s = \Delta u / r$ at the scale r can be written, by a change of variables from Δu to s, as

$$P(s|r) = r p_{\Delta u}(rs) , \tag{10}$$

where $p_{\Delta u}$ is given by Eq. (9) or Eq. (4). The quotation marks for the gradient above reflect the fact that s is the gradient only when r is comparable to the dissipation scale r_D defined by

$$r_D \Delta u(r_D)/\nu = 1 . \tag{11}$$

Note that the average value $\langle r_D \rangle$ of r_D is the Kolmogorov scale η. For the p model, Eq. (11) becomes

$$v(\langle \Delta u_0^2 \rangle^{1/2} L_0/\nu) M^{k/3} (1-M)^{(N-k)/3} 2^{-N} = 1 . \tag{12}$$

Here, we have used the relation $r_D/L_0 = 2^{-N}$, and k, which lies between 0 and N, is randomly picked for the p model from a binomial distribution. Equation (12) and the knowledge of the PDFs of v and k will allow us to compute the probability that the dissipation scale is r_D. Let $P_{r_D}(r)$ be the probability that the dissipation scale is r. The probability density for the gradient can now be written from Eq. (10) as

$$p_s(s) = \int dr\, P(s|r) P_{r_D}(r) , \tag{13}$$

where we have weighted the conditional expectation $P(s|r)$ by the probability that the dissipation scale is r. This step, while rigorous, would necessitate cumbersome computations. We avoid them here by making the simplifying assumption that

$$p_s(s) \sim P(s|r = r_D) . \tag{14}$$

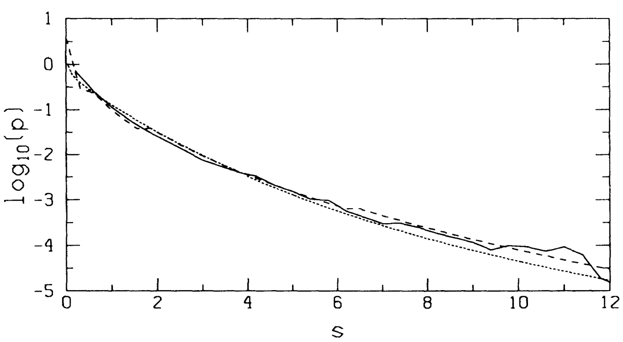

FIG. 3. A comparison between the experiment for the tails of the $p_s(s)$ (solid line) and the theoretical formulas based on the p model (dashed line) and the random β model (dotted line).

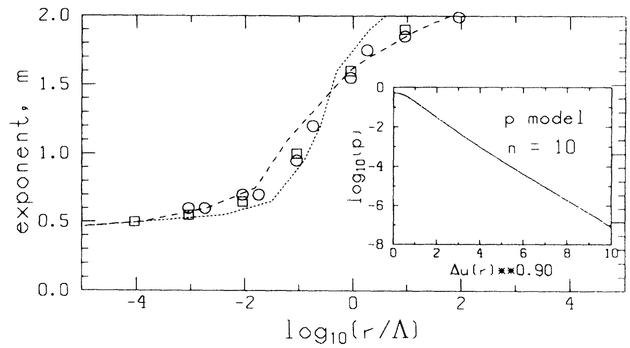

FIG. 4. The stretching exponent m determined empirically for Eqs. (4) and (9), compared with the experimentally determined data of Fig. 1 (dashed line, p model; dotted line, β model). The parameter n is determined arbitrarily by matching the experiment with the theoretical formulas for some r/Λ. Inset: An example in which Eq. (4) for $n = 10$ (corresponding to $r/\Lambda = 0.009$) is well fitted by a stretched exponential with stretching exponent $m = 0.9$.

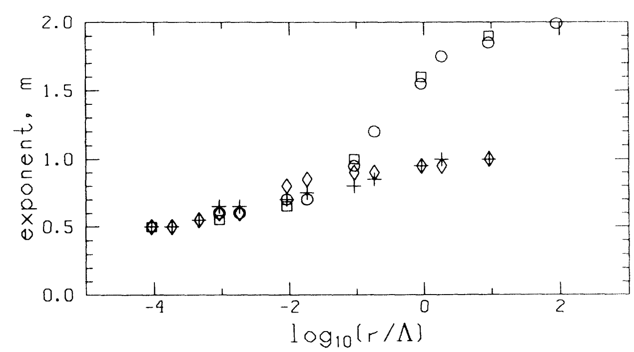

FIG. 5. Demonstration that the PDFs of increments $u_>$, which is the velocity u from which the low-frequency components are removed, tend to an exponential form for large r. This is in contrast to the unfiltered data for which the asymptotic form is nearly Gaussian. The high-pass-filter setting is 30 Hz for + and 9 Hz for ◇. The error bars are comparable to the typical ones shown in Fig. 1.

Noting that $r_D=(\nu/s)^{1/2}$, Eqs. (10) and (14) allow us to express $p_s(s)$ for the p model as

$$p_s(s)\sim\sum_{k=0}^{N(s)}C_k^{N(s)}\frac{1}{M^{k/3}(1-M)^{[N(s)-k]/3}|s|}\exp\left[-\frac{\nu|s|}{2\langle\Delta u_0^2\rangle M^{2k/3}(1-M)^{2[N(s)-k]/3}}\right] \tag{15}$$

(to within a normalization constant). Note that, in Eq. (15), $N(s)=\frac{1}{2}\log_2(L_0^2|s|/\nu)$ and that the approximate sign in Eqs. (14) and (15) is a reminder about their heuristic nature. Using these same assumptions, Benzi *et al.* [9] derived a similar expression for the PDF of the velocity gradient on the basis of the random β model. Figure 3 shows a comparison between experiment and Eq. (15) and an alternative formula due to Benzi *et al.* [9]. The p model yields a slightly better fit to the data.

We now point out that the stretched exponentials approximate Eqs. (4) and (9) even though the latter two are substantially more complex. This is illustrated in the inset to Fig. 4 where Eq. (9) for $n=10$ is plotted against $(\Delta u)^{0.9}$. The tails of the PDF are closely approximated by a stretched exponential, consistent with Figs. 1 and 2. The theoretical expressions have been examined for various n, and the stretching exponents m are determined from the best empirical fits [17]. For the stretched exponentials $p_{\Delta u}(\Delta u)=p_{\Delta u}(0)\exp(-\alpha|\Delta u|^m)$, reasonably good straight lines can be fitted to the plot of $\log\{\log[p_{\Delta u}(0)/p_{\Delta u}(\Delta u)]\}$ vs $\log(|\Delta u|)$ in the range of $|\Delta u|$ between the minimum and maximum of $\sigma_{k;n}$ [see Eqs. (4b) and (9b)]. The exponents so determined are plotted in Fig. 4 by matching with experimental data at one chosen location. The fits are good on the whole.

We now return to the asymmetry of the PDF of $\Delta u(r)$. In an earlier work [7], the velocity signal u was decomposed into its Fourier modes, and the modes below the low-frequency end of the inertial range were eliminated. The remaining Fourier modes were recombined to give the filtered signal, say $u_>$. The increments $\Delta u_>(r)=u_>(x+r)-u_>(x)$ were then found to be symmetrically distributed. By fitting stretched exponentials to the tails of the $p_{\Delta u}(\Delta u_>)$, we found that the exponents were similar to those of the unfiltered data in the dissipation range, but tended towards an exponential for large r (Fig. 5). An inquiry on the exponential behavior of the tails of the PDFs of the temperature field has recently been undertaken [18], but it is not clear whether similar explanations hold for velocity increments in the filtered signal.

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