Scale-Invariant Multiplier Distributions in Turbulence

Ashvin B. Chhabra [(1),(2)] and K. R. Sreenivasan [(1)]
[(1)] *Mason Laboratory, Yale University, New Haven, Connecticut 06520-2159*
[(2)] *Mathematical Disciplines Center and The James Franck Institute, The University of Chicago, Chicago, Illinois 60637*
(Received 4 November 1991)

A family of scale-invariant, base-dependent, multiplier distributions is measured for the turbulence dissipation field in the atmospheric surface layer. The existence of these distributions implies the existence of the more traditional multifractal scaling functions, and we compute both positive and negative parts of the $f(\alpha)$ curve. The results support the conjecture of universality in the scaling properties of small-scale turbulence. A simple cascade model based on the measured multiplier distributions is shown to possess several advantages over previously considered models.

PACS numbers: 47.25.−c, 02.50.+s, 03.40.Gc, 05.45.+b

The multifractal formalism [1–5] [e.g., the $f(\alpha)$ function] compactly describes the scaling properties of measures which arise in a variety of problems such as chaotic dynamical systems [6], diffusion-limited aggregation (DLA) [7], and dissipationlike quantities in fully developed turbulence [8]. The $f(\alpha)$ function is a macroscopic feature of such measures, and a quest for deeper understanding of the underlying physics has led to microscopic descriptions. A successful example is the Feigenbaum scaling function (FSF) [9] appropriate to the onset of chaos in the period-doubling route. Basic to the FSF are the contraction ratios (or multipliers), which describe how distances between the nearest-neighbor iterates scale with increasing levels of refinement. The FSF organizes these multipliers according to the natural time of the system described by the closest return times, and compactly describes the local scaling. The FSF description is much richer than that embodied in $f(\alpha)$, and the latter is easily computable from the former. Further, the $f(\alpha)$ description is degenerate and a variety of scaling functions lead to the same $f(\alpha)$ function [10,11].

Many multifractal measures, such as the spatial distribution of the energy dissipation rate in fully developed turbulence, or the harmonic measure of the DLA structures, display statistical properties that are different from those of deterministic systems. Two such properties are the sample-to-sample fluctuations of the $f(\alpha)$ function, and the existence of negative dimensions—both of which represent the same underlying phenomenon [12,13]. Recall that, if $P_i(\varepsilon)$ is the measure in the ith box of size ε, one can decompose the multifractal measure into interwoven sets of varying singularity strengths α (where $[P_i(\varepsilon)]^q \sim \varepsilon^{q\alpha_i}$), whose fractal dimensions $f(\alpha)$ are defined by $N(\alpha) \sim \varepsilon^{-f(\alpha)}$, $N(\alpha)$ being the number of singularities of strength α. Figure 1 shows how $f(q=2)$ and $\alpha(q=2)$, computed from the turbulence dissipation data using the direct method of Ref. [14], vary from one sample to another; each sample is approximately one macroscale in extent and produces extended and unambiguous scaling. The sample-to-sample fluctuations [15] are much larger than the least-squares errors in calculating $f(q)$ and $\alpha(q)$ from individual samples. In addition, for any given q, $f(q)$ and $\alpha(q)$ from different samples are correlated and fall along a thin band [16].

These phenomena are absent in deterministic processes, and their occurrence reflects an inherently probabilistic dynamics. For such cases, a complete specification of the scaling properties, even at the level of $f(\alpha)$, requires measuring both positive and negative parts of $f(\alpha)$ [12,13,17–20]. Negative $f(\alpha)$ for turbulent flows at moderate Reynolds numbers have already been obtained in Ref. [20] but, for atmospheric flows, their computation by conventional box-counting methods requires an enormous amount of data, involving perhaps several years of data acquisition [12,13,20]. Recently [12,13,21], a

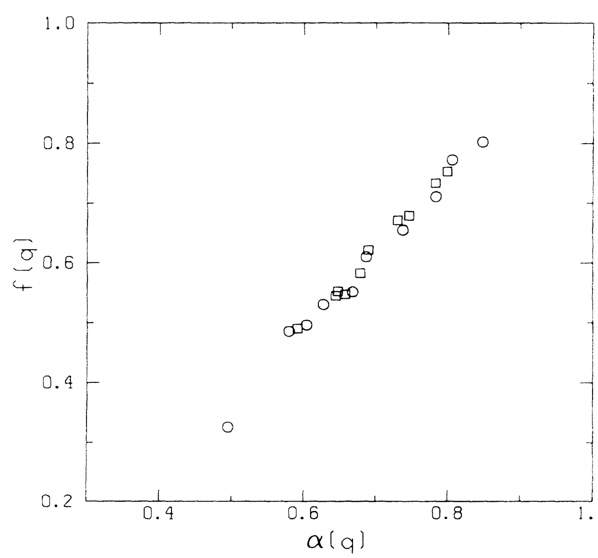

FIG. 1. Sample-to-sample fluctuations of $f(q=2)$ vs $\alpha(q=2)$ for ten samples (7200 points each, roughly equivalent to an integral scale) from an atmospheric boundary layer (squares). The circles are from a simulation of a binary cascade model with multipliers chosen randomly from a triangular distribution. This model is described later in the text (see also Fig. 2).

method based on scale-invariant multiplier distributions has been proposed as being exponentially faster and more accurate than box-counting methods. Utilizing this method, we were able to compute negative dimensions for atmospheric flows as well as a variety of model cascade processes.

This paper has several related objectives. As a continuation of the notion that scale-invariant multiplier distributions are fundamental functions, we compute them for atmospheric turbulence by assuming that the observed scaling could result from cascades of any base (binary, ternary, etc.). Using these distributions, we construct several base-independent functions such as $f(\alpha)$. We then show that the $f(\alpha)$ functions computed from different multiplier distributions agree well with each other, as well as with those computed in moderate-Reynolds-number flows by box-counting methods. Finally, we propose a relatively simple cascade model that reproduces not only the observed positive and negative parts of $f(\alpha)$ and sample-to-sample fluctuations, but also the stretched exponential tails in the probability distribution of turbulence dissipation. The superiority of the multiplier method allows us to compute $f(\alpha)$ with greater accuracy and to assess, for the first time, the universality conjecture about scaling properties in the negative-dimension range.

To compute the scale-invariant multiplier distributions one first constructs a measure which, in the present case, is the energy dissipation rate represented by the square of the derivative of a component of the velocity. The measure is then covered by boxes of uniform size. Each of these boxes is then subdivided into a number of boxes, a, and the ratios of the measures in the original box to those in the smaller sub-boxes are computed. A histogram of these ratios is then $P_a(M_a)$. Figure 2 shows $P_a(M_a)$ obtained by assuming cascades of bases $a=2$, 3, and 5. The larger symbols show an average over steps involving comparisons between boxes of size m and those of size ma, where m ranged from 50 to 1000. The shape of the distribution remains invariant for the inertial range of scales. (For the smallest scales, the distributions have a concave shape. This concavity is related to the divergence of moments [22] and will be discussed elsewhere. For very large boxes, multiplier distributions become flatter, as would be the case for random measures.)

The multiplier distributions $P_a(M_a)$ are more basic than the $f(\alpha)$ function, and were introduced in the context of turbulence by Novikov [1], and measured by Van Atta and Yeh [23]. However, their importance with regard to multifractals in turbulence has subsequently been ignored. One can compute from the multiplier distributions both the positive and negative parts of the $f(\alpha)$ function, i.e., they contain information on the asymptotic scaling properties of a measure and on fluctuations of scaling properties for samples of finite size. In addition, even in instances where high-order moments diverge [2,22], $P_a(M_a)$ remains well defined. Finally, the $f(\alpha)$

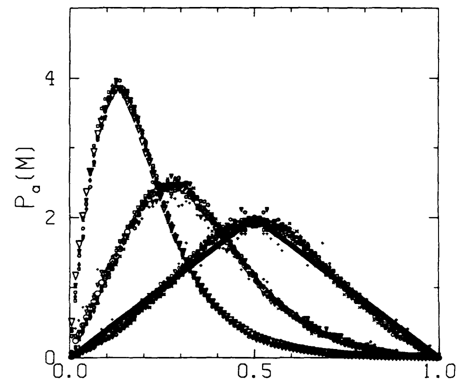

FIG. 2. Multiplier distributions $P_a(M_a)$ for bases (from right to left) $a=2$, 3, and 5. The larger symbols show averages over steps comparing measures in boxes of size m with those in size ma, $50<m<1000$ in units of the Kolmogorov scale. The smaller symbols show the distributions obtained for different values of m (50, 80, 150, 200, 400, and 1000). The solid line is a useful approximation to the $a=2$ distribution.

function may extend over $(-\infty,\infty)$ whereas $P_a(M_a)$ is a compact function defined on $M_a\in[0,1]$. The base dependency of $P_a(M_a)$ can be scaled out because the multiplier distributions corresponding to different bases are related by convolution, provided the multipliers at successive cascades are uncorrelated. With this assumption, several base-independent functions can be constructed from the multiplier distributions.

Consider as an example the Mellin transforms $\mathbf{M}$ of $P_a(M_a)$ and of its convolution $P_a^2(N)$, where N is the product of two multipliers picked according to their probabilities $P_a(M_a)$. Since the Mellin transform of a convolution is the product of the individual Mellin transforms, we have

$$\mathbf{M}\{P^2_{b=a}(M_b)\}=\mathbf{M}\{P_a(M_a)\}\cdot\mathbf{M}\{P_a(M_a)\}$$
$$=[\mathbf{M}\{P_a(M_a)\}]^2 . \qquad (1)$$

It is clear that, in general, the exponent in the last term of Eq. (1) is simply $\log(b)/\log(a)$, the number of times the variable is being convolved. So far, for any two different bases we have

$$[\mathbf{M}\{P_a(M_a)\}]^{\{1/\log(a)\}}=[\mathbf{M}\{P_b(M_b)\}]^{\{1/\log(b)\}} . \qquad (2)$$

On evaluating the Mellin transform and taking logarithms of both sides, we get [1]

$$\frac{\log\langle(M_a)^q\rangle}{\log(a)}=\frac{\log\langle(M_b)^q\rangle}{\log(b)}=-[\tau(q)+D_0] . \qquad (3)$$

The other scaling functions $f(q)$, $\alpha(q)$, and D_q are simply related to $\tau(q)$, and so they too can be easily derived from the multiplier distributions. Further, one can derive equivalent scaling functions by using Laplace or Fourier transforms.

We now compute the $f(\alpha)$ function from the multiplier

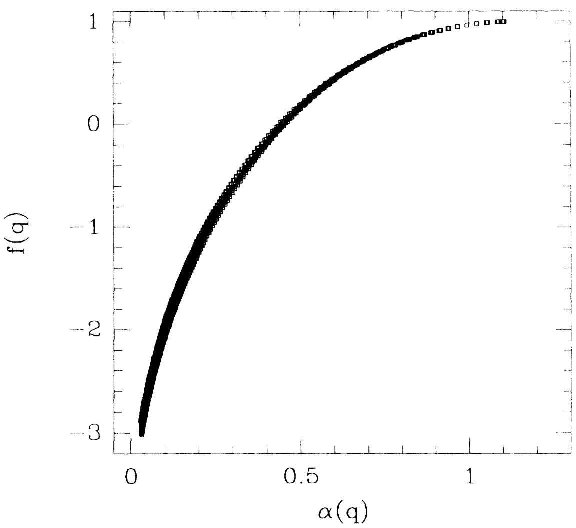

FIG. 3. The $f(\alpha)$ function from the atmospheric data for bases 2, 3, 4, 5, 6, 8, and 10, after the base-dependent prefactor has been removed. The good collapse implies that the assumption of random multiplicative processes is essentially correct, and that no single base is preferred.

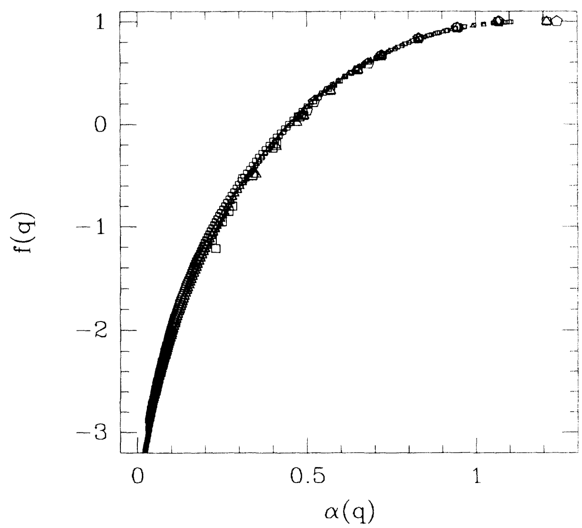

FIG. 4. The $f(\alpha)$ function for the atmospheric data for bases 2 and 10. Also reproduced (from Ref. [20]) are the data from a variety of laboratory flows (larger symbols). The small diamonds are from computations [28] using multipliers from the triangular distribution shown in Fig. 2. The agreement between $f(\alpha)$ functions for atmospheric turbulence and for laboratory flows supports the notion of universality of scaling properties at different Reynolds numbers even of the most rare events.

distributions of Fig. 2, and verify that the multifractal scaling functions derived from different-base multiplier distributions indeed collapse. While doing so, it is important to account for the existence of a prefactor in $\langle (M_a)^q \rangle \sim C(q) a^{\{-\tau(q)-D_0\}}$. From Eq. (3), we have $\tau_a(q)+D_0=\tau(q)-\log[C(q)]/\log(a)+D_0$, but the prefactor can be eliminated by computing $\tau_a(q)$ for two different bases. One can then obtain the correct base-dependent exponents for an arbitrary base. Internal consistency requires that all of them should collapse onto a single curve. Figure 3 shows just such a collapse for bases $a=2$, 3, 4, 5, 6, 8, and 10 (where the prefactor was computed using bases 2 and 4). The collapse strongly indicates that there is no preferred base as far as it concerns the scaling properties.

Finally, Fig. 4 compares the scaling exponents for atmospheric flows with those computed [20] from laboratory data. The latter have been computed using conventional box-counting methods and thus do not assume uncorrelated multiplicative processes. The exponents for atmospheric flows have been computed using the multiplier method which uses just such an assumption. The excellent agreement with conventional box counting (in the range where the latter is capable of yielding exponents) indicates the correctness of our convolution arguments based on uncorrelated multiplicative processes. (Such an approach would fail for the period-doubling attractor because the contraction ratios in that case are highly correlated, thus invalidating the simple convolution arguments used here.) We thus observe a nontrivial and interesting empirical fact that the successive multipliers in cascades, which give rise to the observed intermittency in turbulence, seem to be essentially uncorrelated. The comparison in the negative $f(\alpha)$ region shows excellent agreement between the laboratory and atmospheric scaling properties, supporting the conjecture of universal scaling for even the rare events in the small-scale velocity field in fully developed turbulence.

In the past few years a variety of cascade models such as the log-normal model [24], beta model [25], random beta model [26], and p model [8] have been proposed to mimic intermittency in turbulence. None of these models displays sample-to-sample fluctuations or negative dimensions; nor do they explain the multiplier distributions shown in Fig. 2. One can, however, construct an entire family of simple cascade models which display all of the above features and whose $f(\alpha)$ functions agree well with experiment. The simplest model would be a binary model with multipliers picked randomly from a triangular distribution (shown by the solid line in Fig. 2) which is a rough approximation to the mean distribution for a binary cascade. Figure 4 shows that the $f(\alpha)$ function for this model is in good agreement with other experimental data. As already remarked, this cascade model displays the right behavior with respect to sample-to-sample fluctuations shown by circles in Fig. 1. The model also reproduces the observed [20,27] stretched exponential tails, $P(\varepsilon) \sim \exp\{-\beta(\varepsilon)^{1/2}\}$, in the probability distributions of the energy dissipation rate ε. However, it cannot address issues such as the divergence of high-order moments.

We thank J. Eggers, C. J. Evertsz, R. V. Jensen, P. W. Jones, L. P. Kadanoff, B. B. Mandelbrot, C. Meneveau, M. Nelkin, N. Read, and G. Stolovitzky for useful discussions. A.B.C. acknowledges support from ONR and the Materials Research Laboratory at the University of Chicago. K.R.S. acknowledges support from an AFOSR grant.

[1] E. A. Novikov, Prikl. Mat. Mekh. **35**, 266 (1971) [Appl. Math. Mech. **35**, 231 (1971)]; Phys. Fluids A **2**, 814 (1990). The 1971 paper, which has largely gone unnoticed in the context of multifractals, dealt with cascade models and the importance of multipliers in turbulence. Much of that paper translates easily into the multifractal framework.

[2] B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).

[3] H.G.E. Hentschel and I. Procaccia, Physica (Amsterdam) **8D**, 435 (1983).

[4] U. Frisch and G. Parisi, in *Turbulence and Predictability of Geophysical Fluid Dynamics and Climate Dynamics*, edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, New York, 1985), p. 84.

[5] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[6] J. P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985).

[7] P. Meakin, A. Coniglio, H. E. Stanley, and T. Witten, Phys. Rev. A **34**, 3325 (1986).

[8] C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 797 (1987); R. R. Prasad, C. Meneveau, and K. R. Sreenivasan, Phys. Rev. Lett. **61**, 74 (1988); K. R. Sreenivasan, Annu. Rev. Fluid Mech. **23**, 539 (1991).

[9] M. J. Feigenbaum, J. Stat. Phys. **25**, 669 (1978).

[10] M. J. Feigenbaum, J. Stat. Phys. **46**, 925 (1987).

[11] A. B. Chhabra, R. V. Jensen, and K. R. Sreenivasan, Phys. Rev. A **40**, 4593 (1989).

[12] A. B. Chhabra and K. R. Sreenivasan, Phys. Rev. A **43**, 1114 (1991).

[13] A. B. Chhabra and K. R. Sreenivasan, in *New Perspectives in Turbulence*, edited by L. Sirovich (Springer, Berlin, 1991).

[14] A. B. Chhabra and R. V. Jensen, Phys. Rev. Lett. **62**, 1327 (1989).

[15] The same striking phenomenon has been observed in the scaling properties of DLA (unpublished joint work with C. J. Evertsz).

[16] One may argue that the fluctuations disappear in the limit of infinite levels of refinement. However, even geophysical turbulent flows have no more than about a dozen steps in the cascade. Solar turbulence has about 30. Thus one is always dealing with statistical mechanics of small systems and the thermodynamic limit is strictly never reached.

[17] B. B. Mandelbrot, J. Stat. Phys. **34**, 895 (1984).

[18] M. E. Cates and T. A. Witten, Phys. Rev. A **35**, 1809 (1987).

[19] B. B. Mandelbrot, in *Fractals: Proceedings of the Erice Meeting*, edited by L. Pietronero (Plenum, New York, 1989).

[20] C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).

[21] K. R. Sreenivasan, Proc. R. Soc. London A **434**, 165 (1991).

[22] In general, whenever $P(M)$ falls off slower than exponentially, e.g., as M^{-p}, moments of order less than $-(p+1)$ will diverge. Concave-shaped $P(M)$ will result in divergences for positive p values.

[23] C. W. Van Atta and T. T. Yeh, J. Fluid Mech. **59**, 537 (1973); **71**, 417 (1975).

[24] A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).

[25] U. Frisch, P. L. Sulem, and M. Nelkin, J. Fluid Mech. **87**, 719 (1978).

[26] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984).

[27] P. Kailasnath, K. R. Sreenivasan, and G. Stolovitzky, Phys. Rev. Lett. (to be published).

[28] For these computations, we simulated 720 different realizations of a binary multiplicative process with ten steps in each cascade. We calculated multiplier distributions assuming cascades of different bases, and eliminated the base-dependent prefactor in the same manner as for the turbulence data. We note that the exponents so calculated are somewhat different from those evaluated analytically from triangular multiplicative process with unity prefactor. For the present, we do not understand the proper method of analytically estimating these prefactors or of reconciling these differences completely satisfactorily.