A paradox concerning the extended Stokes series solution for the pressure drop in coiled pipes

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While the extended Stokes series (ESS) solution for the laminar pressure drop in coiled pipes predicts that the friction ratio (the ratio of pressure drop in the coiled pipe to that in the straight pipe of the same flow rate) varies asymptotically as one-fourth power of the Dean number, the existing experimental data largely support a square-root variation; previous boundary layer analyses also predict a square-root variation. The subject of this paper is an examination of this paradox. A detailed review shows that the existing set of experimental data can be grouped into various categories depending on the precise flow conditions. It is shown that none of the existing data satisfies the proper experimental conditions required by the ESS method. New experiments reveal that the pressure drop demanded by the ESS solution can actually be observed if they are expressly designed to satisfy the conditions demanded by the ESS solution. The implication is that the domains of initial conditions appropriate to the boundary layer and ESS solutions are quite different.

I. INTRODUCTION

The flow through curved pipes has long been appreciated for its complexity and richness of detail. At the turn of this century, Eustice[1,2] demonstrated the existence of the secondary flow by the dye injection method. White[3] showed that flow through curved pipes can be maintained as laminar for Reynolds numbers substantially higher than the customary 2300 in straight pipes having inlets designed with no special care. Among other things, this leads to the expectation that an initially turbulent flow in such straight pipes can be stabilized by passing it through a curved section or a coil. This phenomenon has been studied by Taylor,[4] Viswanath *et al.*,[5] and, in somewhat greater detail, by Sreenivasan and Strykowski.[6] For several other interesting aspects of the flow through curved pipes, reference must be made to a detailed review by Berger *et al.*[7] Here, we concern ourselves with a paradox/controversy surrounding *laminar* pressure drop in curved pipes of circular cross section.

The earliest theoretical analysis of Dean[8,9] showed that if the helicity of the coil is neglected, the steady, fully developed, laminar flow through a coiled pipe depends primarily on the dimensionless parameter $\mathrm{Re}(d/D)^{1/2}$, which is now known in its many variants as the Dean number, De; here $\mathrm{Re} = Ud/\nu$ is the Reynolds number, U being the bulk mean speed of the flow, d the pipe diameter, D the coiling diameter, and ν the kinematic viscosity. Dean treated the problem of laminar flow through a loosely coiled pipe as a systematic perturbation of the corresponding flow through a straight pipe, and expressed the ratio of the flux through the curved pipe to that through a straight pipe (of the same diameter under the same pressure gradient) as a series in increasing powers of the Dean number. Alternatively, the more easily measured ratio of the friction drop for flow through the curved pipe to that in the straight pipe for the same flux can be seen as the reciprocal of the flux ratio.

The perturbation series obtained by Dean can be extended to higher order by using a computer to take care of the mounting arithmetic labor. This approach, termed the extended Stokes series (ESS) method,[10,11] is very convenient for analyzing many flows that can be treated mathematically as regular perturbation problems. By analyzing the coefficients of the series one can unravel the analytic structure of the solutions that in turn can be used to further improve the utility of the series. Van Dyke[12] took the ESS approach to analyze the flow through coiled pipes and computer-extended Dean's series to 24 terms. The so-called Domb–Sykes plot and an Euler transformation of the Dean number rendered the series accurate for very large Dean numbers. Van Dyke showed that the friction ratio (that is, the ratio of the friction factor f_c in the curved pipe to that in the straight pipe f_s for the same mass flux) varies asymptotically as

$$f_c/f_s \sim 0.471\ \mathrm{De}^{1/4}. \quad (1)$$

Unfortunately, this result differs from that of the earlier boundary layer analyses in the large Dean number limit, first by Adler,[13] and later by Barua,[14] Mori and Nakayama,[15] Ito,[16] and Smith,[17,18] who all predict that the friction ratio varies as the square root of the Dean number. These results can be represented by the equation

$$f_c/f_s = a\,\mathrm{De}^{1/2} + b, \quad (2)$$

with the constants a and b from different analyses listed in Table I. There is a general agreement among these various results. The finite difference calculations made by Truesdell and Adler[19] and Collins and Dennis[20] also support the half-power dependence of the boundary layer analysis. Although the actual difference between Van Dyke's solution and the boundary layer results is very small for low Dean numbers, it is clear that the asymptotic natures of the two sets of results are quite different in the limit of large Dean number. It is this difference that concerns us here.

One objection to the boundary layer analyses in coiled pipes is that the boundary layer structure is unclear in the flow. In fact, despite the similarity of results from different boundary layer analyses, different authors have interpreted

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TABLE I. Friction ratio for large Dean numbers from boundary layer analyses. Constants a and b in Eq. (2).

Reference	a	b
Adler[13]	0.1064	...
Barua[14]	0.0919	0.5093
Mori and Nakayama[15]	0.1080	0.3513
Ito[16]	0.1033	0.4075

the flow in different ways. This led Van Dyke to dismiss the boundary layer solutions as incorrect; in fact, he also claimed that the earlier finite difference methods[19,20] were inaccurate, but more refined computations by Dennis[21] and Dennis and Ng[22] have weakened this contention.

It seems clear that turning to experiment is helpful, and indeed a large number of experiments on the friction ratio exist in the literature. Figure 1 provides a partial compilation of the results. These (and other similar) results have been correlated by Hasson[23] by the expression (for $30 < \mathrm{De} < 2000$)

$$f_c/f_s = 0.097\ \mathrm{De}^{1/2} + 0.556. \tag{3}$$

This being similar to (2), it has generally been concluded that the available experiments uphold the boundary layer analyses. One is then tempted to dismiss the ESS results—even if formally correct—as being of no consequence; one has in mind that there may be a bifurcation of solutions, and that the ESS solution corresponds to an unobservable branch. Mention may be made of the work of Yang and Keller,[24] which suggests that a hierarchy of bifurcation solutions is possible. Earlier, Dennis and Ng[22] discovered that at large Dean numbers there is a bifurcation of the two vortex solutions, both of which, however, have the same behavior for the friction ratio as the boundary layer solutions. So far as we are aware, these bifurcations have not been observed conclusively, although somewhat similar, but inherently more complex, patterns have been found by Cheng and Yuen[25] in their experiments on heated curved pipes.

Part of our contention is that none of the previous experiments were done under conditions required by the ESS

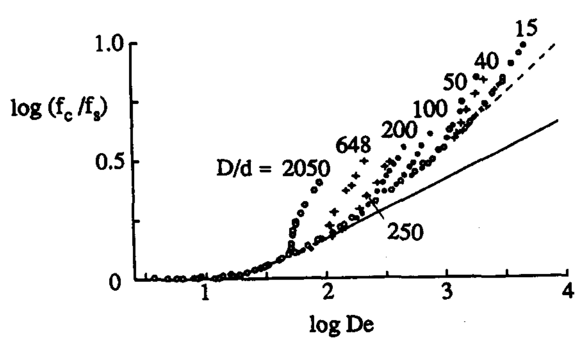

FIG. 1. Experimental friction ratios for various diameter ratios compared with the ESS solution (given by the full line). ○, White[3]; ●, Adler[13]; +, Ito.[29] Dean number as defined in the text. ---, correlation of Hasson.[23] From Van Dyke.[12]

solution. The main points, which we shall justify amply in Sec. II, are twofold. First, the manner in which the high Dean number limit is sought in ESS and the manner it is obtained in most available experiments are different. Second, the flow at the entrance to the coil in the relevant experiments was turbulent. We will show that these features make significant differences to the physical nature of the flow within the coil. In Sec. III, we present new measurements carried out to correspond as closely as possible to the demands of the ESS tenets, and make what we believe is a proper comparison between experiment and theory. Together, these two form the scope of the present paper.

We believe that this effort is worthwhile because, if it turns out that the ESS results for this problem are incorrect definitively, confidence in the entire ESS technique is weakened in contexts beyond the present. This would be unfortunate because the method is attractive for solving the Navier–Stokes equations at sufficiently high Reynolds numbers (the traditional domain of the boundary layer methods), especially for the special class of problems for which the boundary layer structure is unclear or where there is separation. If the conclusion were to be otherwise, the *a priori* implication appears to be that a reexamination of many boundary layer and finite-difference solutions is warranted.

Finally, we note that a similar dichotomy exists between the boundary layer solutions and the ESS results in the case of a straight pipe rotating about an axis perpendicular to the flow direction.[26,27] Experiments would, however, be quite difficult, but the conclusions of the present work should apply with equal force to this problem also.

II. A REEXAMINATION OF THE PAST EXPERIMENTAL DATA

Suppose now that one were to design a new experiment expressly to test the validity of the ESS solution. In this section, we will examine the requirements of such an experiment, and in so doing show that most of the past experiments do not conform to them.

The first requirement is clearly that the flow should be fully developed in the coiled section. For, if that were not so, the friction drop would depend to different degrees on the nature of the flow at the inlet. The hot-wire measurements of Sreenivasan and Strykowski[6] (for $d/D = 0.06$) suggest that this asymptotic state is unlikely to occur before about three turns of the coil, but these measurements did not cover a wide range of Reynolds numbers or diameter ratios. As a somewhat general criterion, we may use one by Yao and Berger,[28] which states that, at large Dean numbers, the entry length l_e for the flow to become fully developed scales as

$$l_e/d = \tfrac{1}{2}e(\mathrm{Re})^{1/2}(d/D)^{-1/4}, \tag{4}$$

where e is a constant weakly dependent on the diameter ratio and the Reynolds number, and lies between 2 and 4. Table II summarizes the diameter ratios and the developing lengths used in the experiments; Fig. 2 shows more graphically that the entry length in a number of experiments is smaller than that required according to (4). A review of the details of previous experiments shows that most of the measurements for diameter ratios greater than $\frac{1}{200}$ were made in the develop-

TABLE II. Summary of experiments in coiled and curved pipes.

Reference	d/D	Re_{cr} [a]	De_{cr} [a]	l_e/d [Eq. (4)] [b]	l_e/d (expt.) [c]	Inlet condition
White[3]	0.0049	2265	50	160–320	1321	
White[d]	0.0200	6300	890	211–422	2000	turbulent
White	0.0667	7750	2000	173–346	450	
Taylor[4]	0.0313	5010	886	168–336	250[e]	turbulent
Taylor	0.054	5830	1350	158–316	not known	
Adler[13]	0.0050	3980	280	237–474	320	
Adler	0.0100	4730	473	217–434	160	probably
Adler[d]	0.0200	5620	795	200–400	80	laminar
Ito[29]	0.0015	8000	315	225–450	2000	
Ito	0.0040	6200	390	313–626	750	
Ito	0.0100	5000	500	224–448	320	laminar
Ito[d]	0.0250	6325	1000	200–400	120	
Ito	0.0610	5265	1300	146–292	50	
Ito	0.0015	2300	90	242–484	2000	
Ito	0.0040	2300	145	191–382	750	turbulent
Ito	0.0610	5265	1300	146–292	50	
Present	0.056	3750	800	120–240	220	laminar
Present[d]	0.0157	6000	750	211–422	600	

[a] The critical Reynolds and Dean numbers listed here correspond to conditions at which a deviation from Hasson's correlation is perceived to occur; as argued in the text, this is not well defined except when the diameter ratio is small. In the case of Adler's experiments, the figures quoted here were given in his paper. The present results correspond to the cessation of a steady laminar state as determined by hot-wire probes at the exit of the pipe.
[b] This represents the length of the developing region according to Eq. (4); the two numbers correspond to $e = 2$ and 4, respectively.
[c] This represents the total length of the curved section used by various authors. White's data are from his paper, while Ito's were obtained assuming one turn of the coil as shown in his Fig. 1. Adler's data correspond to half a turn of the coil; see his Fig. 11.
[d] See text for explanation.
[e] The coil was at least this long; the precise value cannot be ascertained from Taylor's paper.

ing region. For example, the measurements of Adler[13] and Ito[29] were carried out in coils of less than one turn where the flow was clearly not fully developed; in one of White's experiments the "coil" was only a tenth of a turn. The effect of this insufficient developing length on the friction ratio is not obvious *a priori*, but we present in Fig. 3 data from one of our experiments. It is quite clear that the friction factor in this experiment keeps falling with the developing length, thus casting doubt on the validity of claims made on the basis of measurements in developing regions.

The second requirement is that the critical Dean number (that is the Dean number at which the flow ceases to be laminar and steady) should be large because the discrepancy between experiment and the ESS solution is prominent only at high Dean numbers. It is useful to put some numbers on how high the Dean number must be for it to be considered "sufficiently high." At $De = 100$, say, the difference between the available experiments and the ESS results is of the order of 2%, hardly enough to be resolved by any new experiment. At a Dean number of 500, this difference is about 20% which, we shall argue below, is barely large enough. For the case of $d/D = 0.025$ typical of "loosely coiled" pipes, this means a Reynolds number of about 3200; the accuracy of Reynolds number measurement at this value is on the order of $\pm 3\%$. Now, when a circular pipe is bent into a coil, the cross section departs from true circularity and

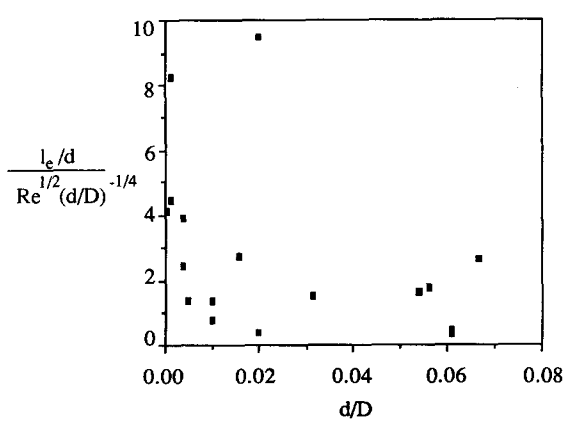

FIG. 2. The entrance length normalized according to Eq. (4). In this normalization, only experiments lying above 2 for the ordinate are fully developed. Data from Table II.

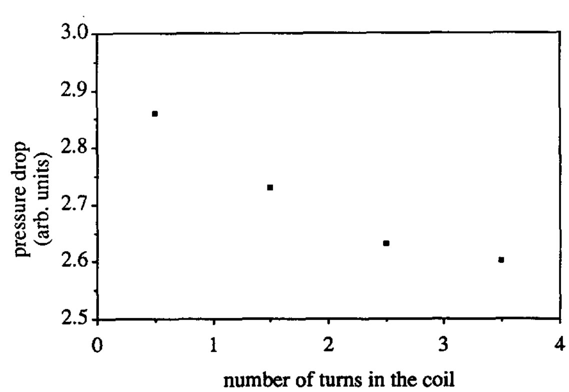

FIG. 3. The dependence of the pressure drop in the developing region of the coiled pipe. The data points correspond to the pressure drop (in arbitrary units) measured between zero and one turn, one and two turns, two and three turns, and three and four turns. $d/D = 0.021$.

becomes slightly elliptical. From our own experience with coiling, as well as a careful reading of the literature, we conclude that a major/minor axis ratio of the order of 1.04 represents a realistic experimental value. As White[3] demonstrated, one effect of ellipticity is to increase the measured friction ratio. Together, these put an uncertainty of the order $\pm 5\%$ on the measured Dean number. A further effect of ellipticity is that the diameter value to be used in the friction factor determination becomes uncertain, thus rendering the overall uncertainty large enough for the new efforts at resolving this issue futile unless the Dean number is at least 500 (preferably larger).

Table II also contains information on the Dean number ranges investigated by various authors. For qualitative purposes, we follow the conventional wisdom (whose limitations we shall soon discuss) and estimate the critical Dean number by the break in the friction ratio curve. A look at the table shows that half the number of experiments do not attain sufficiently high Dean numbers.

The next point of importance is that large Dean numbers must be obtained in a specific way. In the old analysis of Dean the following double limit was sought:

$$Ud/\nu \to \infty, \quad d/D \to 0, \ \text{De fixed.} \tag{5}$$

For tractability in the series expansion, Dean assumed De to be small but Van Dyke extracted the limit for large Dean numbers as mentioned earlier in Sec. I. A comparison with experiments is legitimate only if large Dean numbers are attained *via* the same double limit in the experiments as well. It is seen from Fig. 4 that progressively higher critical Dean numbers are obtained by using increasingly tighter coils, thus possibly violating the requirement of small d/D at some point.

How small must be the diameter ratio for it to be considered sufficiently small? Nunge and Lin[30] suggest that significant d/D effects can occur for ratios at least as small at 0.07. A more detailed effort to answer this question was made by Sreenivasan and Strykowski,[6] whose work was motivated partly by an earlier passing observation of Taylor[4] that dye bands introduced into coiled pipes of diameter ratios 0.054 and 0.031 began to vibrate in an irregular fashion but retained their identity through at least a whole turn of the helix; it was not until an appreciably higher speed was attained that the flow became turbulent and diffusive. Experiments of Sreenivasan and Strykowski not only confirmed this, but also showed that roughly below a diameter ratio of 0.03, a different phenomenon occurred; the dye band, instead of oscillating as Reynolds number was increased, abruptly broke into turbulent patches. Hot-wire measurements made at the exit of 20 turn coils of various diameter ratios less than 0.03 showed that, as the flow Reynolds number was increased continuously, the instantaneous velocity showed an abrupt transformation from the steady laminar state to a turbulent one without any intermediate states that were qualitatively distinct, although details did depend to some extent on the precise entrance conditions to the coil. For tighter coils, the steady laminar state was followed by a roughly periodic laminar state before the onset of turbulence. Whatever the reasons for this different behavior, it was clear that a significant qualitative difference existed between the flow in coils of radius ratio less than about 0.03 and that in coils of radius ratio greater than about 0.03. We shall now proceed to show that this diameter ratio marks the largest value acceptable in the sense of "loose coiling."

A few preliminaries are required to establish this fact. First, we note that, in the past, the flow was determined to be laminar as long as the measured friction ratio did not depart from a curve common to all the experimental data; this is the one to which Hasson fitted the expression given by Eq. (3). As can be seen from Fig. 1, the departure from this common curve was rather sharp for small diameter ratios, thus making this determination quite precise. For increasingly larger diameter ratios, this was not the case. There was in general a gradual deviation from the accepted laminar line, making the determination of the critical Dean number unreliable at best. It is quite possible, as has been later shown by Sreenivasan and Strykowski,[6] that the flow that was thought to be laminar and steady was not necessarily so; independent hot-wire measurements had not been made.

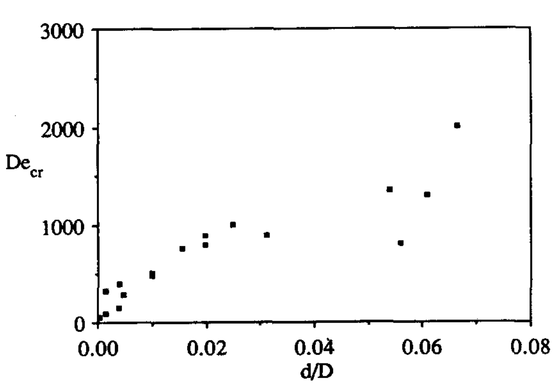

FIG. 4. Figure showing that increasingly higher critical Dean numbers are generally obtained by increasing the diameter ratio. Data from Table II.

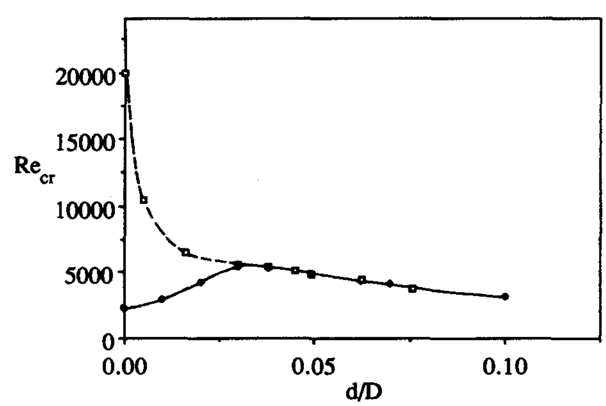

FIG. 5. Critical Reynolds numbers marking the end of the steady laminar state. The upper curve is for a disturbance-free flow at the entrance to the coil (more or less uniform distribution of velocity), while the lower curve is for fully developed turbulent flow at the inlet. The two curves are identical for $d/D > 0.03$.

There are two ways by which the flow can be kept laminar and steady. To see this, we have plotted in Fig. 5 the critical Reynolds number (indicating the loss of steady laminar state) for two cases. In both cases, $d/D = 0$ merely represents conditions just upstream of the coils. The upstream conditions remained the same for all d/D ratios in each of the two experimental sets. For the lower curve, the entrance was preceded by a long straight pipe whose inlet was designed with no special care, and the critical Reynolds number in the straight section was around the customary 2300. At the entrance to the coiled section, the flow was fully developed and of the Poiseuille type. This was assured to be the case by independent measurements in another, nominally identical, straight pipe of equal length. The measurements included velocity profiles in two orthogonal planes and the pressure drop as a function of Reynolds number, which followed the expected trend accurately. (The relevant data, as well as a detailed description of how the Reynolds numbers and pressure drop were measured are best relegated to Sec. III.) For the upper curve, the entrance straight section was effectively nonexistent, but the inlet was designed with sufficient care so that the flow at the entrance to the coil remained steady and laminar until after a Reynolds number of 20 000 was reached.

The following observations were made. For the lower curve, the critical Reynolds number in the coil increases with increasing d/D until a value of about 0.03 is reached. At 0.03, the critical Reynolds number is around 5200. This means that the entrance flow, even if it were turbulent, will be rendered laminar as long as the Reynolds number is below 5200; the curvature has a stabilizing effect. As d/D increases past 0.03, the critical Reynolds number in the coil starts to decrease, and falls off roughly like $(d/D)^{-0.5}$. The Reynolds number range for which the curvature is stabilizing will diminish, although it is stabilizing even at a d/D of 0.1 (the largest value up to which our measurements extend), this range being only 1000 or so. Thus relaminarization is one way of attaining a steady laminar flow in the coiled section. The other way is to start with a laminar flow that is sufficiently "disturbance-free" at the coil entrance so that it stays laminar right through the entire coil. As can be seen from the upper curve, the curvature effects are quite the reverse, and all entrance laminar flows are destabilized beyond some d/D. That is, the flow that is laminar at the entrance to the coil will lose its stability at some lower Reynolds number once inside the coil.

The interesting observation is that for $d/D > 0.03$ the two curves collapse; in fact, our measurements with several more inlet Reynolds numbers (not presented here to avoid cluttering of the data) have confirmed that for all inlet conditions the critical Reynolds numbers collapse on to this common curve. For smaller diameter ratios, however, different inlet Reynolds numbers will lead to different curves.

The data of Fig. 5 can be replotted in terms of the critical Dean number (Fig. 6). It is again immediately apparent that for $d/D > 0.03$, the largest Dean number up to which the flow can be maintained laminar is about 1000, and is unique for different initial conditions. On the other hand, for smaller diameter ratios, the critical Dean number could be

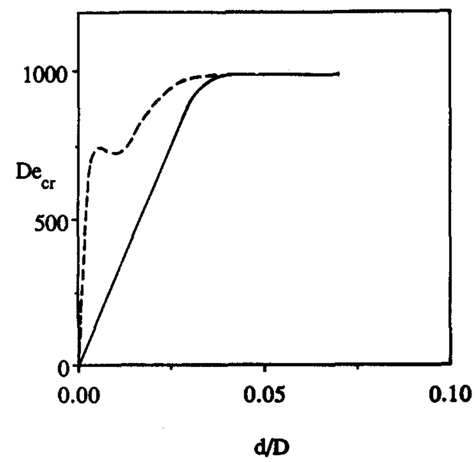

FIG. 6. Critical Dean numbers marking the end of the steady laminar state. The two curves correspond to those in Fig. 5.

quite different for different initial conditions—both smaller and possibly larger than 1000.

The main point to be made is that for small diameter ratios, curvature is destabilizing for certain entrance conditions and stabilizing for others. Inasmuch as the same curvature has two different effects depending on the initial conditions, the state of the flow in the coil must be different for different initial conditions, and so we surmise that the pressure drop could be different also. As d/D increases, the differences between the two states become increasingly smaller until they vanish at around 0.03. For tighter coils, one inlet condition is like another. It follows from experience with any one flow in a tight coil that all the other flows in tight coils do not follow the ESS solution. On the other hand, for $d/D < 0.03$, since there are two possible steady laminar states, it may well be that one of them behaves like the ESS solution.

It is useful to recapitulate our thoughts so far. The only past experiments of likely interest to us here are those in which the flow is established to be laminar and steady, with enough developing length, the coil diameter ratio not exceeding 0.03, and Dean numbers substantially larger than 500. There is very limited data satsifying all these requirements (see Table II and Fig. 2). Within this subset, there are two further subclassifications based on possible entrance conditions: one of them laminar and the other turbulent. In the former (designated as class L below), the flow remains laminar everywhere in the coil, while in the latter, the steady laminar state observed in the coil is attained by the upstream relaminarization of the entering turbulent flow (class T).

Among the available experiments, to class T belongs one of the experiments of White (marked by the superscript d in Table II). In this experiment, the measured friction drop does not follow the ESS solution (Fig. 7). It is thus clear that class T flows are not of the ESS type. It is worth noting that one of Ito's experiments (marked by the superscript d in Table II), and possibly one of Adler's (also marked by the superscript d), satisfy all the criteria of class L except that they do not have enough developing length. Ito's own plots show that data from his experiment also fall close to Hasson's correlation. This should not surprise us, however, be-

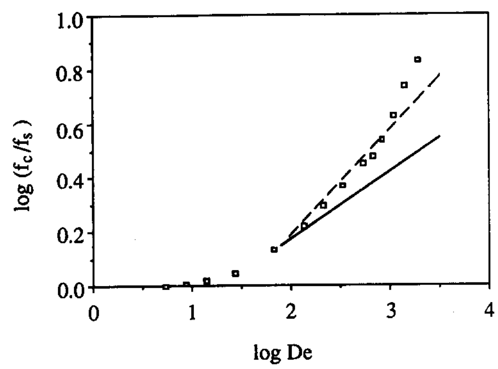

FIG. 7. The friction ratio in one of White's experiments satisfying all the requirements of the ESS solution except for the inlet flow being turbulent. Note that it departs from the ESS solution. The symbols are from White's paper, but it should be mentioned that White specifically notes that they were not the actual experimental data points. —, ESS solution; – –, Hasson's correlation.

cause of the very limited entrance length. If we account for this by assuming that Ito's flow behaves similar to the one in Fig. 3, and rescale the measured pressure drop accordingly, Ito's data fall a fair bit below Hasson's correlation (Fig. 8). Another less important point is that friction ratios in Ito's experiments (and in others starting with White) are obtained by normalizing the experimentally obtained friction factors in the curved pipes by the theoretical value of 64/Re appropriate to fully developed Poiseuille flow. This, of course, is a valid procedure in principle, but it is internally consistent to determine both friction factors from measurement. A case in point would be when both the straight and coiled sections are slightly rough (for instance). The measured friction factor for the straight pipes in Ito's experiments are in fact higher than the theoretical value of 64/Re; see Fig. 3 from Ito.[29] If we form the friction ratios using Ito's experimentally obtained friction factor in the straight pipe, his data fall even closer to the ESS solution. Similar conclusions hold also for Adler's data.

We readily admit that these *ad hoc* "correction" procedures are not to be taken very seriously, especially because the "correction" resulting from insufficient developing length is likely to be of different magnitudes at different

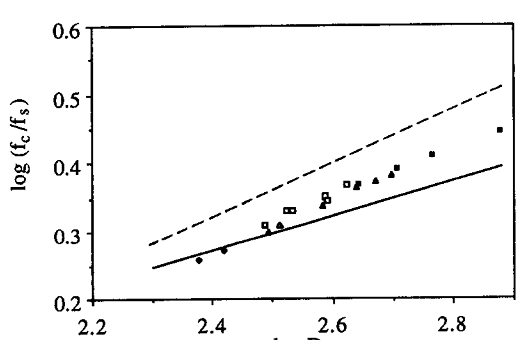

FIG. 8. Friction ratio obtained by Adler[13] and Ito.[29] Since the measured friction drop in the coiled section of these experiments is in the developing region, the plotted data have been modified assuming that an extrapolation as per Fig. 3 is appropriate. ♦, $d/D = 1/200$ (Adler); ■, $d/D = 1/50$ (Adler); □, $d/D = 1/100$ (Adler); △, $d/D = 1/100$ (Ito). The two lines represent the ESS solution and Hasson's correlation.

Dean numbers, whereas we have used as guidance for correction data from only one experiment. The outcome, however, suggests that no existing flow really belongs to class L. We report the results of such an experiment in the next section.

III. EXPERIMENTS AND RESULTS

The qualifications of an ideal experiment of class L are the following. The disturbance level of the incoming flow must be small enough for the flow to remain laminar for relatively high Reynolds numbers in coils of diameter ratio less than 0.03 in such a way that usefully large values of the critical Dean number result in the coiled section. The coiled section must be several turns in extent. Further, to insure reasonably good circularity of the pipe section in the coil, metal pipes should be used; for detailed measurements, a fairly large diameter of the pipe would be desirable. Although the present experiments do not ideally match these requirements (because of practical constraints), they satisfied them reasonably well. They are described below.

We chose copper pipes of 15.75 mm internal diameter, this size being large enough to make detailed cross-sectional measurements. The longest single piece of pipe available to us (about 10 m) was slightly longer than 600 diameters. Since we intended to create smooth initial conditions, joining pipes of several sections was rejected as an alternative for making a longer pipe. Also forced on us from practical constraints of coil manufacturing was a diameter ratio of 0.0157 (the coil diameter of about 1 m). Care was taken to avoid the distortion of the circular cross section during coiling and the ellipticity was well within 4%.

For pressure drop measurements static pressure holes were made as follows. Thin copper rods were soldered on the outside circumference of the pipe at various distances from the inlet and 0.5 mm diam holes were drilled through them all the way to the inner wall of the pipe. By patiently scraping the inner surface by soft steel wool attached to the end of a flexible rod, any possible burrs on the inside wall were removed. The soldered copper rods were sticking out about 5 mm above the outside circumference of the pipe, and were connected to an M. K. S. Baratron Unit with a 10 Torr differential pressure head; the pressure readings obtained were accurate to better than 0.1%. The pressure taps were fixed on the exposed outer wall in the plane of symmetry of the coil. Although the static pressure at any axial section varied along the circumference of the pipe, the pressure drop measured between pressure taps mounted at two different axial locations (at the same azimuthal location) was the same, irrespective of the precise azimuthal position.

Compressed air from two large storage tanks (capacity 25 m^3) was passed through a rotometer into a large settling chamber, a honeycomb and several screens at designated places in the settling chamber, ending in a smooth nozzle with a contraction ratio of 25:1.

With these experimental conditions, we could attain a critical Reynolds number of at least 20 000 in a straight pipe, and about 6000 in the coiled section of diameter ratio of 0.0157. This resulted in a critical Dean number of about 750 in the coiled section. This was not as high as we would have wished, but it is still large enough (see Sec. II) to distinguish whether measurements conform to the ESS solution or Hasson's correlation.

Measuring the friction drop in the coiled pipe is now a relatively simple matter, but two issues had to be settled first. One was the determination of the flow Reynolds number accurately. The other was the choice of the normalizing pressure drop in the straight pipe. As mentioned earlier, either the experimental pressure drop or the theoretical pressure drop would be acceptable if we could make sure that they were the same for a straight pipe of the same stock. For these purposes, we made detailed experiments in an independent straight pipe chosen from the same stock as the experimental pipe. Pitot probes in conjunction with the Baratron unit were also used to measure the velocity profiles of the straight pipe flow, and it was ensured that the velocity profile in two orthogonal planes followed the parabolic profile quite well (Fig. 9).

Rotometer calibration for flow Reynolds numbers was done with great care, and by several methods; that indeed was the most time consuming part of the experiment. The first method used the integrated velocity profiles at the exit of the straight pipe; this was relatively straightforward since the profiles were symmetric. (Note that the inlet conditions to the straight pipe were smooth enough for the flow to be laminar up to a Reynolds number of 20 000. It was, however, not fully developed and parabolic at the exit section for Reynolds numbers above about 6000, as can be seen from the friction factor measurements presented in Fig. 10.) The second procedure consisted of measuring the (nominally uniform) velocity distribution at the entrance to the straight pipe, with corrections incorporated for boundary layer growth on the walls of the contraction nozzle upstream. Finally, the rotometer was calibrated by collecting the air flowing through the pipe into a large chamber for a fixed amount of time and measuring the mass of the air collected. The flow

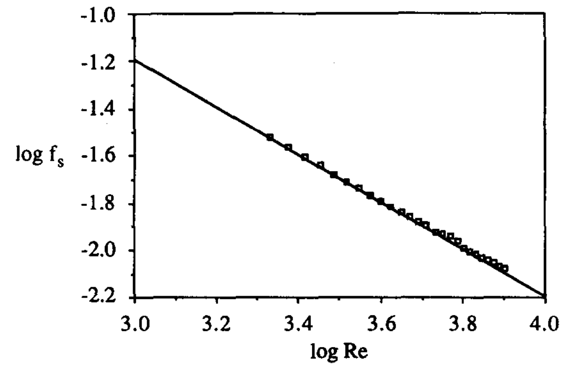

FIG. 10. Friction factor in the straight pipe. —, theory = 64/Re.

Reynolds numbers from 2000 to 20 000 were thus known with high accuracy.

The final measurements now consisted of the pressure drops in the curved and straight pipes for the same rotometer reading; as indicated already, it would have made no difference, for the Reynolds number of interest to us here, if we chose 64/Re for the friction factor in the straight pipe, instead of the experimental values. For the curved pipe, measurements were made between the second and the third coils; this represented about 450 and 600 diameters into the coil which, according to the criteria of both Sreenivasan and Strykowski[6] and Yao and Berger,[28] seemed large enough for the flow to be fully developed before reaching the third coil. The friction ratio thus obtained for flow between two and three coils is shown in Fig. 11. It can be seen that the measurements fall below the curve predicted by the boundary layer theory and are close to the ESS solution. Further, making reference to Fig. 3, since even the third coil is not far enough downstream of the entrance for the incoming flow to be fully developed, it is possible that the asymptotic values of the friction ratio could be marginally lower, perhaps falling right on the ESS result. The limitation on the available contiguous piece of pipe restricted the number of the coils to three, but it is clear that an even longer pipe would have been desirable. One should also remember[3] that the effect of slight

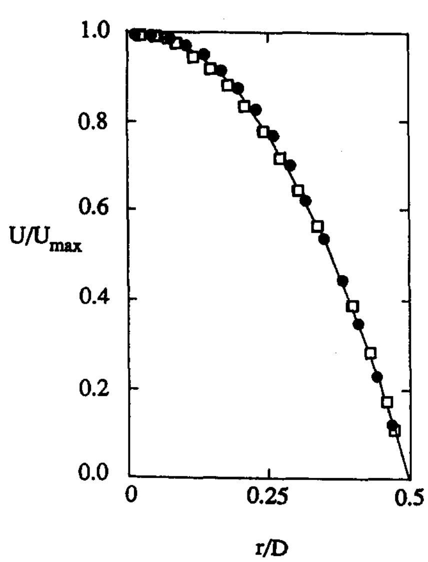

FIG. 9. Typical velocity profiles measured at the exit of a straight pipe of length 600 diameters in two orthogonal planes represented by two different symbols. Full line is the parabolic distribution; Re = 3300.

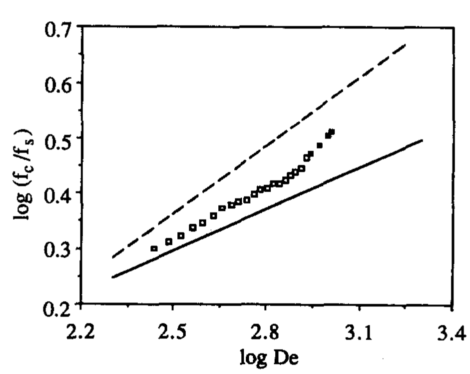

FIG. 11. Friction factor measured in the coiled pipe between two and three turns. Open and closed symbols correspond, respectively, to laminar and turbulent states in the coil. $d/D = 0.0157$. —, ESS theory; – –, Hasson's correlation.

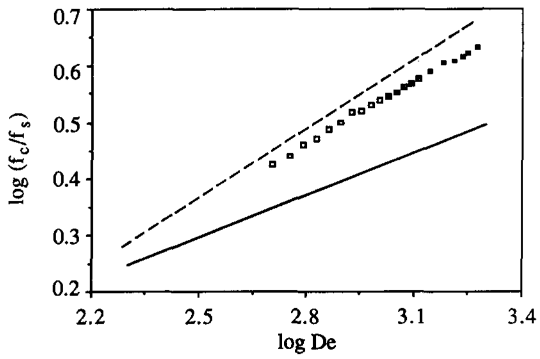

FIG. 12. Friction ratios in a tightly coiled pipe measured between three and four turns (the flow was fully developed). Dotted symbols represent unsteady laminar flow, and filled symbols represent the turbulent regime; $d/D = 0.056$. The two lines are as in Fig. 11.

but inevitable ellipticity in the cross section of the coiled pipe would be to increase the measured pressure drop above that in a coil of truly circular cross section.

Figure 12 shows the friction ratio measured between three and four turns in the tightly coiled pipe; and, as suspected earlier, the transitional and turbulent friction ratios follow the laminar regime rather smoothly and thus the historical method of identifying the flow as being turbulent using the deviation of the friction ratios from Hasson's correlation is clearly incorrect.

IV. CONCLUSIONS

We have shown that there is a set of conditions for which the pressure drop demanded by the ESS solution is actually observed. The conditions are that the inlet flow must be laminar, the diameter ratio must be less than 0.03, the critical Dean number must be higher than about 500, and the entrance length must be large enough for the flow to be fully developed. Of these, the last two are easily understood, and need no special mention here.

The condition on d/D ratio is necessitated by Dean's double limit of $d/D \to 0$ and $\mathrm{Re} \to \infty$. For tighter coils, it appears that the laminar flow is not always steady. At present, it is fair to say that the unsteady laminar flow through coiled pipes is not well understood. This unsteady laminar motion of the flow introduces another important parameter $\alpha = (\omega/\nu)^{0.5} d/2$, where ω is the frequency of the oscillations and ν is the kinematic viscosity, and is called the Womersley parameter in the blood flow literature, where the understanding of pulsatile flow through curved pipes (blood vessels such as aorta, for instance) is of great importance. This parameter (based on the dominant frequency in the flow) is typically about 20 in the present experiments. The theoretical analysis for oscillatory flow in curved pipes is usually based on the assumptions of small Dean numbers and vanishingly small diameter ratios,[31] although boundary layer-type analysis has been used for large Dean numbers.[18] However, these analyses are not useful for the present situation because the diameter ratio could not be considered vanishingly small.

If the inlet flow is turbulent, it can be rendered laminar in the coil up to a Reynolds number of the order of 5000, or equivalently, using the largest admissible diameter ratio, a Dean number of the order 1000. This is large enough for the differences between the ESS solution and Hasson's correlation to show up. However, it is clear from the measurements of White and some of our own that the ESS pressure drop is not observed in this case.

On the other hand, if the entry flow is laminar and remains so through the entire coil, it appears that the pressure drop follows the prediction of the ESS solution. In our experiments, the entrance laminar flow was more or less of the top hat type, and it is not clear what changes would occur if this profile were fully developed and parabolic. Obviously, satisfactory resolution of this question requires another experiment with a long inlet straight pipe upstream of the coil; at a Reynolds number of the order of 6000; conventional wisdom (see, for example, Schlichting[32]) requires at least an upstream section of 400 diameters. Since our chief concern here was to establish whether the ESS solution holds at least for some set of conditions, we did not feel compelled to pursue the matter much further.

One final remark is in order. In spite of the present contributions, we are quite aware that the matter is not resolved completely satisfactorily, at least because the discrepancy between the Dennis–Ng calculations[22] and the ESS solutions of Van Dyke,[12] both of which are based on the same physical model, remains unexplained. It appears to us that the calculation of merely the pressure drop ratio is inadequate to put the matter to complete rest; we therefore suggest that the numerical solutions in the future must look beyond the friction ratio. This suggestion is easier made than carried out, however.

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