Reprinted from June 1982, Vol. 104, Journal of Fluids Engineering

K. R. Sreenivasan
Applied Mechanics,
Mason Laboratory,
Yale University,
New Haven, CT 06520

R. Narasimha
Department of Aeronautical Engineering,
Indian Institute of Science,
Bangalore, India

Equilibrium Parameters for Two-Dimensional Turbulent Wakes

The parameters characterizing a plane turbulent wake in its equilibrium state of development are determined through careful experiment and analysis.

1 Introduction

It is useful to consider the development of two-dimensional turbulent wakes in terms of the parameters

$$\Delta \equiv \delta(x\theta)^{-1/2} \tag{1}$$

and

$$W \equiv (w_0/U)\,(x/\theta)^{1/2} \tag{2}$$

defined by Narasimha and Prabhu [1]. (The notation is explained in the inset to Fig. 1: θ is the momentum thickness, see equation (3) below.) It is expected that in the limit of small defect ratios $w_0/U \rightarrow 0$, the above two parameters tend asymptotically to certain constant values, say Δ^* and W^*, which are universal numbers characteristic of the plane equilibrium wake. (Following [1], we define an equilibrium wake as one in which the mean velocity and turbulent stresses exhibit similarity with *identical* length and velocity scales.)

As experiments can only be conducted at finite w_0/U, it is clear that an accurate determination of Δ^* and W^* may call for extrapolation of measured data to the limit $w_0/U = 0$. Because it was incidental to their work, Narasimha and Prabhu [1] did not attempt such an extrapolation; a preliminary estimate suggested that the numbers quoted by them may require a correction whose magnitude could not be assessed because the measurements were not sufficiently extensive. Other data on wakes, surveyed elsewhere [2, 3] do not permit a definitive determination of the parameters, because of too much scatter or two-dimensional momentum imbalance, or because w_0/U was not sufficiently small. It was therefore thought worthwhile to conduct carefully a new series of experiments with the sole objective of obtaining reliable values for Δ^* and W^*. The primary objective of this work is thus the accurate determination of the equilibrium wake parameters Δ^* and W^* through independent measurements and analysis.

2 The Experiments

Measurements were made in an open-circuit suction-type wind tunnel with a contraction ratio of about 10, and a test-section of about 30 cm square and 4.27 m long. Less than 1.5 percent variation in wind speed along the test section was attained by applying suitable divergence for the boundary layer growth. More details of the wind tunnel can be found in [4]. A two-dimensional wake was created behind a twin plate configuration (see inset to Fig. 1) at a freestream velocity of 21.3 ms^{-1}. The freestream turbulence level at this speed was about 0.15 percent.

It was determined that in the region of measurement the wake was in equilibrium. Although it is known ([3]; [5], Chapter 7) that the wake behind a circular cylinder requires a streamwise distance of as much as 1000 diameters to attain the equilibrium state, the twin-plate wake generator used here seems to be efficient in producing equilibrium wake in much shorter distances (of the order of 200 θ). Detailed measurements [4] of the root-mean-square streamwise and normal velocity fluctuations, as well as of the Reynolds shear stress, confirm this conclusion.

All mean velocity measurements were made with a pitot-static tube. Constant current hot-wire measurements wtih adequate frequency compensation showed that the maximum value of u'/U (where u' is the root-mean-square streamwise velocity fluctuation) ranged from about 4 percent at the closest measuring station ($x/\theta \simeq 130$) to about 1.6 percent at

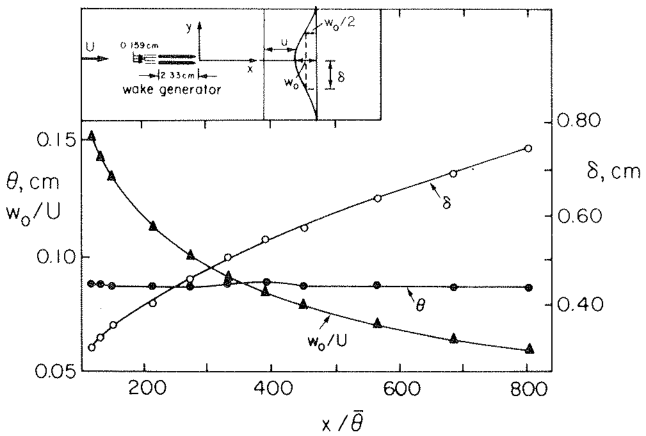

Fig. 1 Variation with the streamwise distance of the center-line wake-defect ratio w_0/U (▲), momentum thickness θ (•) and the half-defect thickness δ (o). The inset shows the wake generator and describes the notation. Average momentum thickness $\bar{\theta}$ = 0.874 mm.

Contributed by the Fluids Engineering Division for publication in the JOURNAL OF FLUIDS ENGINEERING. Manuscript received by the Fluids Engineering Division, January 26, 1981.

the farthest measuring station ($x/\theta \simeq 810$). No corrections possibly necessitated by these turbulence levels were attempted for the pitot-static measurements; simple estimates with the use of Bernoulli's equation were not higher than about 0.4 percent. The wake profiles were found to be very closely symmetric, but all parameters were evaluated using the complete velocity profile rather than one half of it.

The data on the normalized centerline wake defect w_0/U and the wake half-thickness δ are shown plotted in Fig. 1. Also shown is the momentum thickness θ evaluated here as

$$\theta = \int_{-\infty}^{+\infty} \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \int_{-\infty}^{+\infty} \frac{w}{U}\left(1-\frac{w}{U}\right)dy, \qquad (3)$$

$w(y)$ being the defect velocity distribution. It is seen that θ is a constant to within ± 1 percent (except at $x = 35.08$ cm, where the departure from the mean is 1.7 percent). The mean value $\bar{\theta}$ of θ is 0.874 mm (with a standard deviation of 0.007 mm). There is therefore no need to make any convergence corrections of the type described by Prabhu and Narasimha [6].

For the analysis to be made in Section 3, we shall need the parameters I_1 and I_2, where

$$I_n = \int_{-\infty}^{+\infty} (w/w_0)^n d(y/\delta), n = 1, 2. \qquad (4)$$

The values of I_1 and I_2 obtained from measured velocity profiles at various stations are shown in the upper part of Fig. 2. There is no discernible trend with w_0/U over the range covered in the experiments; this result is to be expected if the defect velocity profiles show similarity. The mean values and the 95 percent confidence intervals are:

$$I_1 = 2.061 \pm 0.010, \quad I_2 = 1.505 \pm 0.020. \qquad (5)$$

3 Analysis

From equation (3) and the definitions of I_1 and I_2, it follows that

$$(W\Delta)^{-1} = U\theta/w_0\delta = I_1 - (w_0/U)I_2. \qquad (6)$$

Measured values of the quantity $U\theta/w_0\delta$, also plotted against w_0/U in Fig. 2, follow equation (6) (with I_1, I_2 from equation (5)) quite closely. This result is, of course, not surprising, because it serves only to confirm the well-known fact that wake-defect velocity profiles are indeed self-similar to a very good approximation as long as w_0/U is not large.

The self-preserving solution for the development of a shallow (i.e., linear) plane constant-pressure wake may be written [1] as

$$\frac{U^2\delta^2}{M^2} = \frac{1}{U^2 {w_0}^2} = \frac{2K_1}{M}\frac{x}{U} + K_2, \qquad (7)$$

where K_1 and K_2 are constants and

$$M = U^2 w_0 \delta = U^3 (w_0\delta/U\theta)\theta. \qquad (8)$$

Substituting for M from equation (8), dividing throughout by x and noting from equation (6) that in the limit of vanishing defect $U\theta/w_0\delta \rightarrow I_1$, equation (7) can be reduced to asymptotic statements of the form

$$\Delta \equiv \delta(x\theta)^{-1/2} = \Delta^* + o(1), \qquad (9)$$

$$W \equiv (w_0/U)(x/\theta)^{1/2} = W^* + o(1). \qquad (10)$$

It is not possible to estimate precisely the $o(1)$ terms in equations (9) and (10) on the basis of linear theory alone. However, equation (6) suggests that we should expect corrections of $O(w_0/U)$; Townsend ([5], p. 137) has argued, on the basis of the energy equation, that a term of $O(x^{1/2})$ should appear on the right of equation (7), which implies a term of $O(w_0/U)$ on the right of equations (9) and (10). It is therefore a reasonable assumption to put

$$\Delta = \Delta^* + \alpha(w_0/U) + o(w_0/U), \qquad (11)$$

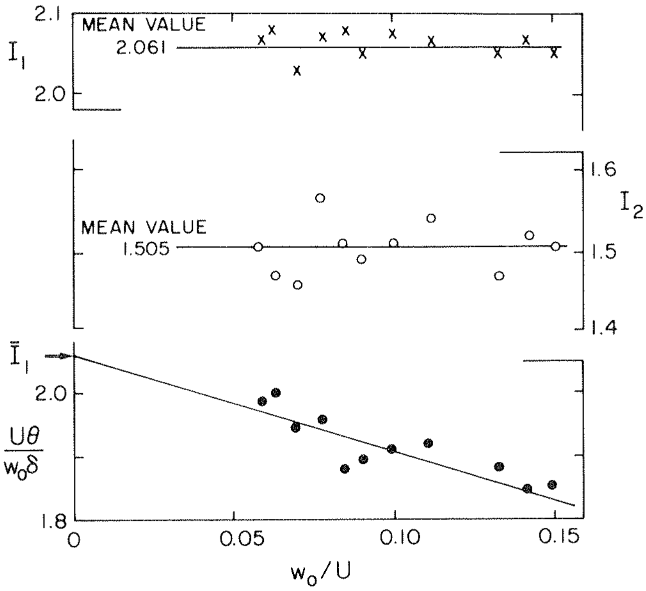

Fig. 2 Variation of the flow parameters I_1, I_2, and $U\theta/w_0\delta$ with the wake-defect ratio. The mean values of I_1 and I_2 are 2.061 and 1.505, respectively. The straight line in the lowermost part of the figure is given by $I_1 - I_2(w_0/U)$, with $I_1 = 2.061$ and $I_2 = 1.505$.

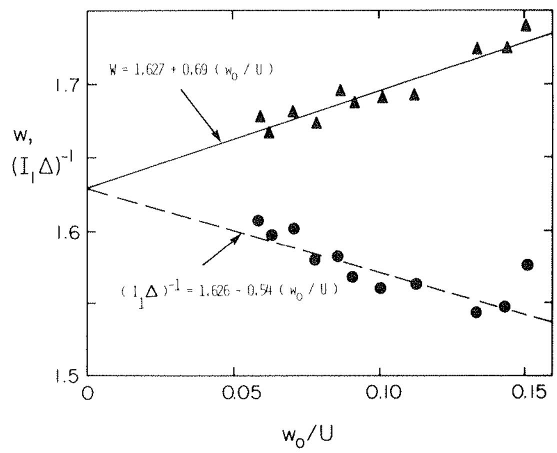

Fig. 3 Variation of the measured quantities W (▲) and $(I_1\Delta)^{-1}$ (•) with the wake defect ratio. ——, $W = 1.627 + 0.69 (w_0/U)$; – –, $(I_1\Delta)^{-1} = 1.626 - 0.54 (w_0/U)$.

$$W = W^* + \beta(w_0/U) + o(w_0/U), \qquad (12)$$

where α and β are some constants. It follows from equations (11) and (12) that

$$(W\Delta)^{-1} = (W^*\Delta^*)^{-1}[1 - (\alpha/\Delta^* + \beta/W^*)w_0/U] + o(w_0/U), \qquad (13)$$

and from comparison with equation (6) that

$$(W^*\Delta^*)^{-1} = I_1, \qquad (14)$$

$$\alpha/\Delta^* + \beta/W^* = I_2/I_1. \qquad (15)$$

For later convenience, we also note that

$$(I_1\Delta)^{-1} = W^* - \alpha(W^*/\Delta^*)(w_0/U) + o(w_0/U). \qquad (16)$$

As experiments give us I_1 and I_2 and a series of data points for $W(x)$ and $\Delta(x)$, we can determine the four parameters W^*, Δ^*, α and β, by making a best fit of equations (11) and (12) or equivalently of equations (16) and (12) to the experimental data. This leaves equations (14) and (15) to be used as checks. Between them, equations (12), (14), (15), and (16) provide such strong constraints and checks that the parameters can be determined to a very good accuracy.

Figure 3 shows W and $(I_1\Delta)^{-1}$ plotted against w_0/U.

According to equations (12) and (16), these data should lie on straight lines intersecting on the axis $w_0/U= 0$, with the common intercept equal to W^*. This feature is the obvious advantage of using equation (16) instead of equation (11). Linear regression analysis gives

$$W = 1.627 + 0.688\ (w_0/U) \tag{17}$$

and

$$(I_1\Delta)^{-1} \simeq 1.626 - 0.542(w_0/U); \tag{18}$$

from equations (18) and (5), it follows that

$$\Delta \simeq 0.298 + 0.099(w_0/U). \tag{19}$$

As expected, the intercepts of equations (17) and (18) on the (w_0/U) axis are both practically the same, lending great force to the present analysis.

As further checks on accuracy (see equations (14) and (15)), we see from equations (17)-(19) that

$$(W^*\Delta^*)^{-1} \simeq 2.063, \tag{20}$$

in excellent agreement with the mean value from direct measurements of I_1. Similarly,

$$\frac{\alpha}{\Delta^*} + \frac{\beta}{W^*} \simeq 0.755, \tag{21}$$

which agrees with the ratio $I_2/I_1 \simeq 0.73$ to within about 3½ percent.

4 Discussion and Conclusion

It may be considered that the discrepancy of 3½ percent in this last consistency check is too large, considering the precision with which the equilibrium parameters have been deduced earlier in Section 3. However, there is a relatively large uncertainty associated with the numerical values of α and β because they show nontrivial sensitivity to small perturbations in the measured wake parameters. Thus, the numerical values of α and β are not as reliable as those of I_1, I_2, W^*, and Δ^*; on the other hand, the accuracy of W^* and Δ^* is adequately proved by the check concerning $(W^*\Delta^*)^{-1}$. However, in view of the other uncertainties such as possible errors due to finite turbulence level in pitot-static tube measurements, etc., it is considered that the equilibrium wake parameters (except perhaps I_1) cannot here be quoted to an accuracy better than about 1 in 100. The best estimates from the present measurements and analysis therefore are:

$$\Delta^* = 0.30 \pm 0.005, W^* = 1.63 \pm 0.02,$$
$$I_1 = 2.06 \pm 0.01, I_2 = 1.51 \pm 0.02.$$

In each case, the error estimates correspond to 95 percent confidence intervals.

These values are quite close to the values[1] quoted by Narasimha and Prabhu [1] ($\Delta^* = 0.298$, $W^* = 1.595$, $I_1 = 2.05$, and $I_2 = 1.50$). In the sense that the wakes of Narasimha and Prabhu [1] had finite defect, this close agreement may be somewhat of a coincidence. Nevertheless, the contribution of this note has not been in giving new values for equilibrium parameters but in confirming, through careful analysis and experiment, the suggestion of reference [1].

Acknowledgment

We are indebted to Dr. Prabhu for many useful discussions.

[1] Due to a misprint in [1] the value quoted for W^{*2} appears as 2.34 instead of 2.54.

References

1 Narasimha, R., and Prabhu, A., "Equilibrium and Relaxation in Turbulent Wakes," *Journal of Fluid Mechanics*, Vol. 54, 1972, pp. 1-17.
2 Sreenivasan, K. R., "Data on Constant-Pressure Two-Dimensional Turbulent Wakes," *Rep. 74 FM5*, Department of Aeronautical Engineering, Indian Institute of Science, Bangalore, 1974.
3 Sreenivasan, K. R., "Approach to Self-Preservation in Plane Turbulent Wakes," *A.I.A.A. Journal*, Vol. 19, 1981, pp. 1365-1367.
4 Prabhu, A., "Non-equilibrium Wake Flows," *Ph.D. thesis*, Department of Aeronautical Engineering, Indian Institute of Science, Bangalore, 1971.
5 Townsend, A. A., *The Structure of Turbulent Shear Flow*, 1st ed., University Press, Cambridge, U.K., 1956.
6 Prabhu, A., and Narasimha, R., "Turbulent Non-Equilibrium Wakes," *Journal of Fluid Mechanics*, Vol. 54, 1972, pp. 19-38.

DISCUSSION

P. Freymuth[2]

This discusser finds the accurate determination of parameters Δ^* and W^* for the two-dimensional wake and how they are asymptotically reached a very valuable contribution to the knowledge of equilibrium wakes. The paper also paves the way to improving the accuracy for corresponding results for the axisymmetric wake and possibly also for 2-d and axisymmetric equilibrium jets, along similar lines.

This discusser hopes that parameters Δ^* and W^* turn out to be truly universal, i.e., that they also apply to the 2-d "standard wake" generated behind a circular cylinder and wonders whether asymptotic trends are also the same. He sees little difficulty in repeating the experiments by the authors in the wake behind a circular cylinder. For their double plate wake generator the authors claim approximate similarity for $x = 200\,\theta$ and beyond with $\theta = 0.087$ cm according to Fig. 1. In terms of the diameter of the wake generator (outer distance between the two plates, $D = 0.159$ cm as suggested by the inset to Fig. 1) one finds $x \simeq 110\,D$. Uberoi and Freymuth [D1] and Freymuth and Uberoi [D2] find approximate similarity for the cylinder wake beyond $x = 100\,D$ and more accurately beyond $x = 400D$. Repeating the experiments for the cylinder seems promising and would give additional stature to the results by the authors.

This discusser would like to mention that the momentum thickness θ can be expressed in terms of the drag coefficient C_D by

$$\theta = 0.5\ \mathrm{D} \cdot C_D \tag{1}$$

Equation (1) allows formulation of results in terms of $D\,C_D$ instead of θ, which is a popular alternative.

Additional References

D1 Uberoi, M. S., and Freymuth, P., "Spectra of Turbulence in Wakes Behind Circular Cylinders," *Phys. Fluids*, Vol. 12, 1969, pp. 1359-1363.
D2 Freymuth, P., and Uberoi, M. S., "Structure of Temperature Fluctuations in the Turbulent Wake Behind a Heated Cylinder," *Phys. Fluids*, Vol. 14, 1971, pp. 2574-2580.

[2] Professor, Department of Aerospace Engineering Sciences, University of Colorado, Boulder, Colo. 80309.

Authors' Closure

Two points should be made in response to Professor Freymuth's discussion of our paper. First, the parameters Δ and W have indeed been measured in two-dimensional wakes behind various bodies such as the circular cylinder, square cylinder with a diagonal parallel as well as perpendicular to the upstream flow and flat plates parallel to the stream. These results, reported elsewhere [3], show that Δ and W in all the wakes approach the asymptotic values Δ^* and W^* determined here. However, the equilibrium state characterized by Δ^* and W^* is approached at different rates and through different routes in different wakes. The twin-plate wake shows the fastest approach with W and Δ approaching their respective asymptotic values monotonically. For the circular cylinder, the process of attaining equilibrium takes a larger distance in terms either of x/θ, x/D or simply of the defect ratio w_0/U; further, the parameters Δ and W show complex non-monotonic behaviors. Both these factors are undoubtedly related to the long memory of the energetic large eddies generated in the shear layers separating from the circular cylinder; it appears that the mean strain field immediately behind the twin-plate generator promotes the disintegration of energetic large eddies, resulting in a simpler behavior further downstream. The value of the parameters Δ^* and W^* lies in the fact that there indeed appears to be unique equilibrium state, even though at earlier stages the wakes behind different bodies are dynamically different.

Secondly, in a true equilibrium state, the mean velocity as well as the Reynolds shear stress distributions should show self-similar shapes when scaled on the *same* velocity and length scales, as emphasized in [1]. As Professor Freymuth points out, this takes at least as much as 400 diameters (and probably longer [5]) for a circular cylinder. For the twin-plane wake, on the other hand, measurements [3, 4] show that this distance is about a quarter that for the cylinder (in terms of D defined by Professor Freymuth). Thus, the use of a twin-plate has a decided advantage in terms of practical considerations such as the finite length of the wind-tunnel test-section, especially when one is interested in observing further development after subjecting the turbulence structure of an equilibrium wake to an external distortion. This was indeed the reason why Narasimha and Prabhu [1] developed the twin-plate configuration for this study of nonequilbrium wakes.