RAPID DISTORTION OF AXISYMMETRIC TURBULENCE

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ABSTRACT

For a special but non-trivial case of homogeneous and initially axisymmetric turbulence, the change in component energies due to a sudden but otherwise arbitrary three-dimensional distortion is found in closed form.

1. INTRODUCTION

THE effect of an externally imposed arbitrary distortion on homogeneous and initially isotropic turbulence was worked out in detail by Batchelor and Proudman[1]—hereafter B and P—on the assumption that the distortion was rapid enough to justify the neglect of inertia and viscous forces. These results have been found useful in a variety of flow situations (*see, e.g.*, Townsend[2]) although their applicability is limited by the assumption of isotropy, which is rarely (if ever) found in practice. On the other hand, axisymmetric turbulence is much more common, especially in wind tunnels. This note presents results for the change in individual turbulent energy components produced by rapid but otherwise arbitrary three-dimensional distortion, for a non-trivial case of initially axisymmetric turbulence in which a pair of independent functions characterizing the spectrum of turbulence is assumed to depend only on the wave-number magnitude. (For the interpretation of such special fields *see* Chandrasekhar[3].) Earlier analyses of the axisymmetric problem are generally incomplete, and sometimes erroneous, as will be discussed in detail elsewhere.

For brevity, familiarity with B and P is assumed here.

2. FORMULATION

The general second order two-point axisymmetric spectrum tensor is known (*see, e.g.*, Batchelor[4]) to be

$$\phi_{ij}(\underset{\sim}{k}) = A_1 k_i k_j + A_2 \delta_{ij} - A_3 \lambda_i \lambda_j - A_4 \lambda_i k_j - A_5 \lambda_j k_i \qquad (1)$$

where $\underset{\sim}{\lambda}$ is a unit vector along the axis of symmetry and each of the five functions A_i depends in general on $k = |\underset{\sim}{k}|$ and $k_1 = \underset{\sim}{k} \cdot \underset{\sim}{\lambda}$. Symmetry in the indices i, j, and the equation of continuity, require that only two of the functions A_i are independent, so that (1) may be written (with a prime denoting conditions before distortion) as

$$\phi'_{11}(\underset{\sim}{k}) = (k^2 - k_1^2)\, F(k, k_1)$$

$$\phi'_{nn}(k) = (k^2 - k_n^2)(F + G) - k_1^2\, G(k, k_1), \quad n = 2, 3, \qquad (2)$$

where $G = -A_3/k^2$ and $F = -(A_1 + A_3)$.

Closely following B and P, the expressions for the spectra after distortion (indicated by double primes) are obtained as:

$$\phi''_{11}(\underset{\sim}{\chi}) = \frac{Fk^2}{e_1^2 \chi^4}\left[k_1^2\left(\frac{k_2^2}{e_2^4} + \frac{k_3^2}{e_3^4}\right) + \left(\chi^2 - \frac{k_1^2}{e_1^2}\right)^2\right] + \frac{G k_1^2 k_2^2 k_3^2}{\chi^4}\left(\frac{e_2^2}{e_3^2} + \frac{e_3^2}{e_2^2} - 2\right) \qquad (3)$$

$$\phi''_{nn}(\underset{\sim}{\chi}) = \frac{Fk^2}{e_n^2 \chi^4}\left[k_n^2\left(\frac{k_1^2}{e_1^4} + e_1^4 e_n^4 (k^2 - k_1^2 - k_n^2)\right) + \left(\chi^2 - \frac{k_n^2}{e_n^2}\right)^2\right] + \frac{G(k^2 - k_1^2 - k_n^2)}{\chi^4} \times \left[e_1^4 e_n^2 (k^2 - k_1^2)^2 + k_1^2 \left\{\frac{k_1^2}{e_1^4 e_n^2} + 2(k^2 - k_1^2)\right\}\right]. \qquad (4)$$

3. ARBITRARY DISTORTION

The ratio of the energy component in the direction of symmetry is given by

$$\mu_1 = \frac{\iiint \phi''_{11}(\underset{\sim}{\chi})\, D\underset{\sim}{\chi}}{\iiint \phi'_{11}(\underset{\sim}{k})\, D\underset{\sim}{k}} \qquad (5)$$

where $D\underset{\sim}{\chi}$ and $D\underset{\sim}{k} = dk\, dS(k)$ are volume elements in χ and k spaces respectively.

We first note, from (2), that

$$\int_0^\infty k^4 F dk = \frac{3}{8\pi}\bar{u}_1^2$$

and

$$\int_0^\infty k^4 G dk = -\frac{3}{4\pi}\left(\frac{R-1}{R}\right)\bar{u}_1^2, \qquad (5a)$$

where

$$\mathrm{R} = \bar{u}_1^2/\bar{u}_n^2$$

$$\bar{u}_1^2 = \int_{-\infty}^{\infty} \phi'_{11}(\underset{\sim}{k})\, \mathrm{D}\underset{\sim}{k}, \text{ etc.}$$

Using (2) and (4) in (5), we get

$$\mu_1 = \frac{3}{8\pi e_1^2 k^2} \int\int \chi^{-4} \left[k_1^2\left(\frac{k_2^2}{e_2^4} + \frac{k_3^2}{e_3^4}\right) + \left(\chi^2 - \frac{k_1^2}{e_1^2}\right)^2\right] dS(k)$$
$$- \frac{3}{4\pi k^4} \frac{\mathrm{R}-1}{\mathrm{R}} \left(\frac{e_2^2}{e_3^2} + \frac{e_3^2}{e_2^2} - 2\right) \times \int\int k_1^2 k_2^2 k_3^2 \chi^{-4} dS(k) \qquad (6)$$

where, using $(5a)$, integrations with respect to k have effectively been cancelled out. Thus all that needs to be known of the initial state is the ratio R; the exact forms of $\phi_1'(k_1)$ and $\phi_n'(k_1)$ are irrelevant.

The second term in (6), giving the modifications due to axisymmetry, vanishes if (a) the turbulence is isotropic (R = 1) or if (b) the distortion is axisymmetric $(e_2 = e_3)$. After integration on the azimuthal angle has been performed, one gets

$$\mu_1 = \bar{\mu}_1 - \frac{3}{2}\frac{\mathrm{R}-1}{\mathrm{R}}\left(\frac{e_2^2}{e_3^2} + \frac{e_3^2}{e_2^2} - 2\right) \times \int_0^{\pi/2} \frac{\cos^2\theta \sin^5\theta\, d\theta}{(\mathrm{B}_2\mathrm{B}_3)^{1/2}(\mathrm{B}_2^{1/2} + \mathrm{B}_3^{1/2})} \qquad (7)$$

where

$$\mathrm{B}_i = \cos^2\theta\left(\frac{1}{e_1^2} - \frac{1}{e_i^2}\right) + \frac{1}{e_i^2},$$

and the overbar symbol here denotes corresponding isotropic quantities.

The integral in (7) can always be expressed in terms of elliptic integrals of the first and second kind, $\mathrm{F}(\phi, m)$ and $\mathrm{E}(\phi, m)$[5]. Thus, if $e_3 \geqslant e_2 \geqslant e_1$,

$$\mu_1 = \bar{\mu}_1 - \frac{3(\mathrm{R}-1)}{2e_1^4\mathrm{R}} \frac{1}{(e_2^2 - e_3^2)^2} \left[\left(\frac{e_2}{e_3} + \frac{e_3}{e_2}\right) \times \left\{\frac{e_3}{e_2} - a\mathrm{E}(\alpha, q)\right\} + \frac{1}{3}\left(\frac{e_3}{e_2}b^2 + \frac{e_2}{e_3}a^2\right) \times \left\{2a\frac{e_3^2 + e_2^2 - 2e_1^2}{e_1^2}\mathrm{E}(\alpha, q) - ab^2\mathrm{F}(\alpha, q) + \frac{e_3}{e_2}(1 - 2a^2 - b^2)\right\} - \frac{2}{3}\right]$$

where

$$a^2 = \frac{e_3^2 - e_1^2}{e_1^2},\quad b^2 = \frac{e_2^2 - e_1^2}{e_1^2},\quad q^2 = \frac{a^2}{b^2},$$

$$\alpha = \arctan(b^{-1}).$$

Similarly,

$$\mu_n = \mathrm{R}\bar{\mu}_n - \frac{3(\mathrm{R}-1)}{2} e_1 \int_0^{\pi/2} \sin^3\theta \times \left[1 + \frac{\left(\frac{1}{e_1^2 e_n^2} - e_n^2\right)}{e_n^2 + e_1^2\tan^2\theta}\right]^{\frac{1}{2}} d\theta. \qquad (8)$$

This can be integrated to give, for instance,

$$\mu_2 = \mathrm{R}\bar{\mu}_2 - \frac{3(\mathrm{R}-1)}{2} e_1 \left[\frac{1+3b^2}{3ab^2}\mathrm{F}(\beta, q) - \frac{2(a^2+b^2)}{3ab^4}\mathrm{E}(\beta, q) + \frac{2e_1^2 - e_3^2}{ab^2 e_1^2}\mathrm{E}(\alpha, q) - \frac{a^2b^2 - 2a^2 - b^2}{3b^4} - \frac{e_2}{e_3}\frac{2e_1^2 - e_3^2}{b^2 e_1^2}\right]$$

where $\beta = \arctan(a^{-1})$.

If the e_i are ordered differently, standard transformations can be used to express μ_1 and μ_n in terms of elliptic integrals with real and positive modulus and argument. In practice, however, numerical calculation of the integrals in (7) and (8) is easier. Finally, writing (7) and (8) in a slightly different way as

$$\mu_1 = \bar{\mu}_1 - \frac{\mathrm{R}-1}{\mathrm{R}}(\Delta\mu_1)$$

and

$$\mu_n = \bar{\mu}_n - (\mathrm{R}-1)(\Delta\mu_n),$$

where $\Delta\mu_1$ and $\Delta\mu_n$ are independent of R, it can be shown that $\Delta\mu_1$ is always positive whereas $\Delta\mu_n$ takes both positive and negative values. Figures 1 and 2 show the 'correction terms' $\Delta\mu_1$ and $\Delta\mu_n$

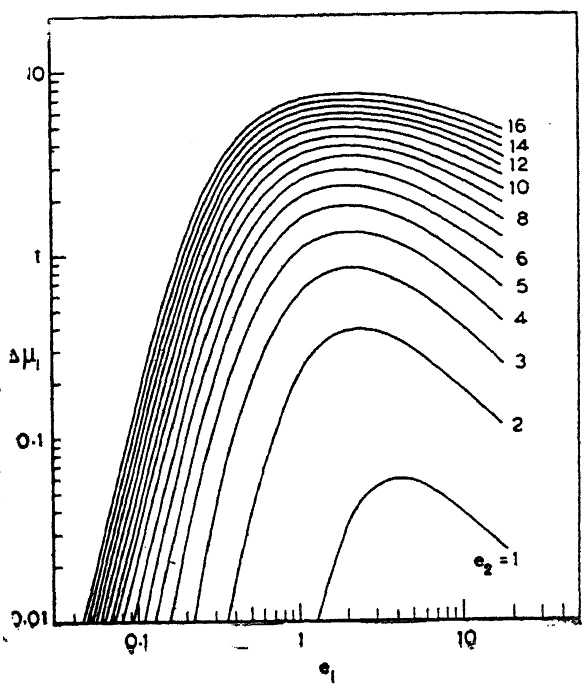

FIG. 1

plotted in a convenient form as a function only of distortion geometry. $\bar{\mu}_1$ and $\bar{\mu}_n$ can be obtained from the analysis of B and P.

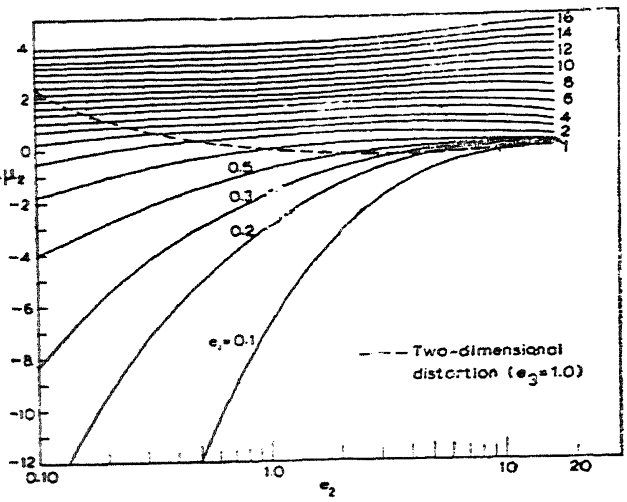

FIG. 2

4. SOME SPECIAL CASES

(*a*) Axisymmetric Distortion : Here $e_2 = e_3 = e_1^{-\frac{1}{2}}$. It follows from (7) and (8) that

$$\mu_1 = \bar{\mu}_1$$

and

$$\mu_n = \bar{\mu}_n - (\mathrm{R} - 1)(e_1 - \bar{\mu}_n).$$

for large e_1, $\mu_1 \rightarrow e_1^{-2}(ln\, 4e_1^3 - 1)$

and

$$\mu_n \rightarrow e_1(1 - \tfrac{1}{4}\mathrm{R}).$$

The limitations of the above analysis for $\mathrm{R} > 4$ are obvious.

(*b*) Large two-dimensional distortions : If $e_2 = 1$, $e_1 = e_3^{-1} > 1$ one gets

$$\mu_1 \approx \bar{\mu}_1 - \frac{\mathrm{R} - 1}{\mathrm{R}}(e_1 - e_1^{-1})^2 \times \left\{\frac{1}{2e_1(e_1 - 1)^2} + 0(e_1^{-2})\right\}$$

and

$$\mu_n \approx \mathrm{R}\bar{\mu}_n - e_1\left[1 - (-1)^n \frac{e_1^2 - 1}{4e_1^4} + 0(e_1^{-6})\right] \times (\mathrm{R} - 1).$$

To order e_1^{-1}, however, these results tend to the case (*a*).

(*c*) Small two-dimensional distortions : If $e_2 = 1$ and $(e_1 - 1)$ is small and positive,

$$\mu_1 \approx \bar{\mu}_1 - \frac{4}{35}\mathrm{B}\frac{1}{e_1(e_1 + 1)^2}$$

$$\mu_n \approx \bar{\mu}_n - (\mathrm{R} - 1)\left[e_1 - (-1)^n \frac{1 - e_1^{-2}}{10e_1}\right]$$

where

$$\mathrm{B} = e_1^{-2} + e_1^{-1} - 2.$$

5. CONCLUSION

This analysis represents a rational alternative to the use of isotropic theory for the case of small departures from isotropy (as in flow behind grids), if only for 'weak' turbulence.

Further details and assessment of results are available in Sreenivasan[6].

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